ANTIBOUND STATES OF THE PION–PION SYSTEM

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The purpose of this paper is to show that dispersion relations strongly indicate that there is an antibound state in the $I = 0$ S state of the reaction $\pi^+ + \pi^- \to \pi^+ + \pi^-$, close to threshold (corresponding perhaps to the ABC "particle") \(^1\). A possible antibound state occurs in the $I = 2$ S wave, but much further from the threshold: none is indicated for the $I = 1$ amplitude.

The simplest demonstration uses zero momentum transfer invariant amplitudes:

$$A(s, t) \equiv \begin{bmatrix} A^0 \\ A^1 \\ A^2 \end{bmatrix}$$

(1)

$s$ and $t$ are the Mandelstam variables and the superscripts are the isospin labels. Writing $A^I_1(s)$ for the partial wave amplitudes, one has

$$A^I_1(4, 0) = A^I_0(4) = a^I_1 (\text{say}) \quad I = 0, 2$$

$$A^1_1(4, 0) = 0$$

(2)

where $a_0$ and $a_2$ are the S wave scattering lengths. The values of these amplitudes at the crossed threshold, $s = 0$, are given by the crossing relation

$$A(s, t) = \beta A(4-s-t, t)$$

(3)

$$\beta = \begin{bmatrix} \frac{1}{\sqrt{3}} & -1 & \frac{5}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{5}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

(4)

Hence

$$A^0(0, 0) = \frac{1}{3} (a_0 + 5a_2)$$

$$A^1(0, 0) = -\frac{1}{3} (a_0 - \frac{5}{2} a_2)$$

$$A^2(0, 0) = \frac{1}{3} (a_0 + \frac{1}{2} a_2)$$

(5)
Most dynamical analyses give positive values for $a_0$ and $a_2^2$, so that $A^o$ and $A^2$ are positive at $s = 0$ and $s = 4$.

The condition for an antibound state in a partial wave $I, l$ is that a pole should appear between $s = 0$ and $s = 4$ on the second Riemann sheet. The function on the second sheet is defined by

$$B^I_l (s) = A^I_l (s) \left[ 1 - 2 \left( \frac{4 - s}{s} \right)^l A^I_l (s) \right]^{-l}$$

Here the surd is defined to be cut for $(-\infty < s \leq 0)$ and $(4 \leq s < \infty)$ and to be positive for $(0 < s < 4)$ on sheet I. Then the condition for a pole of $B^I_l (s)$ at $s = m^2$ is

$$A^I_l (m^2) = \frac{i}{2} \left[ \frac{m^2}{4 - m^2} \right]^{l/2}$$

If one makes the simple approximation

$$A^I (s, o) = A^I (s) . \text{ for } I = 0, 2$$

$$A^I (s, o) = 3 A^I (s)$$

where partial waves for $l = 2, 3, \ldots$ have been neglected, it follows that the condition for an antibound state for $I = 0$, $l = 0$ is

$$A^o (m^2, o) = \frac{i}{2} \left( \frac{m^2}{4 - m^2} \right)^{1/2}$$

Since the right-hand side of this equation is $0$ at $m^2 = 0$ and $+\infty$ at $m^2 = 4$, it follows from the fact that $A^o (s, 0)$ is positive at $s = 0$ and $s = 4$ that (9) must be satisfied for some $m$, which is therefore
the mass of an antibound state. A similar consideration applies to the 
I = 2, \ l = 0 amplitude. On the other hand \( A^1(s,0) \) is 0 at \( s = 4 \) 
and from (5) could be positive or negative at \( s = 0 \). A more sensitive 
test is necessary for this amplitude.

A further approximation will be made in order to derive a simple 
relation between the mass of the antibound state and the \( I = 0, \ l = 0 \) 
scattering length. The following relation is exact at the symmetry point 
\( s = \frac{4}{2} = t \) :

\[
A^0(\frac{s}{2}, t) = \frac{5}{2} A^2(s, t)
\]  
(10)

It will be assumed to be approximately true at \( s = 4, \ t = 0 \). Then

\[
\alpha_0 \sim \frac{5}{2} \alpha_2
\]  
(11)

Thus (5) gives

\[
A^0(\sigma, \phi) = \alpha_0 = A^0(4, \phi)
\]
\[
A^1(\sigma, \phi) = 0
\]
\[
A^2(\sigma, \phi) = \alpha_2 = A^2(4, \phi)
\]  
(12)

Hence \( A^0 \) and \( A^2 \) are separately equal at the direct and crossed thresh-
olds \( s = 4 \) and \( s = 0 \). They will be assumed to be approximately constant 
throughout \( 0 < s < 4 \) (this cannot be exactly true, of course).

Then (7) is easily solved, to give

\[
m^2_I = \frac{16 a^2_I}{1 + 4 a^2_I} \quad I = 0, 2.
\]  
(13)

together with (11).
Here \( m_0 \) and \( m_2 \) are the masses of the \( I = 0 \) and 2 antibound states. Fig. 1 shows \( m_0^2 \) and \( m_2^2 \) plotted against \( a_o \). Thus, for \( a_o = 1 \), \( m_0^2 = 3.2 \) and \( m_2^2 = 1.6 \); and for \( a_o = 1.5 \) (which is entirely possible), \( m_0^2 = 3.6 \) and \( m_2^2 = 2.4 \). (The unit is the pion mass.)

Hence one expects an \( I = 0 \) antibound state quite close to the normal threshold. This will be identified with the ABD "particle", and can be invoked to "explain" the large \( I = 0 \) scattering length. There may also be an \( I = 2 \) antibound state, but this is further from the threshold, and would have a smaller effect on low energy scattering. Moreover, it is more likely to be strongly affected by the approximations in this treatment; and an accurate analysis may even remove it.

The analysis can be repeated with the quantity \( F(s) = \frac{A_1(s,0)}{4-s} \) in order to determine whether there is an \( I = 1 \) antibound state. Defining the \( P \) wave scattering length by

\[
A_1 = \lim_{q \to 0^+} \frac{A_1^1(4(q^2+1))}{q^2}
\]

one has

\[
F(4) = -\frac{3}{4} A_1
\]

and

\[
F(0) = \frac{1}{2} A_1
\]

with the same approximations as before. Now \( A_1 \) is known to be small and positive, of the order of \( 0.05 \). On the other hand, the antibound state condition (7) reads, for \( F(s) \),

\[
F(m^2) = \frac{3}{2} \frac{1}{\sqrt{m^2(4-m^2)}}
\]

and the minimum of the modulus of this function is \( \frac{3}{4} \). Accordingly (16) is probably never satisfied and there is no \( P \) wave antibound state.
In conclusion a few words can be said about improving these approximations. Firstly, instead of assuming constancy in \(0 < s < 4\), the amplitudes can be evaluated in this interval by using a dispersion relation. Secondly, combinations of the amplitudes and their first derivatives can be considered which eliminate the \(D\) wave identically. Calculations embodying these improvements will be published later.

A still better approximation would probably be to use the partial wave dispersion relations directly to find the solution of (7).
REFERENCES


2) B.H. Bransden and J.W. Moffat, Nuovo Cimento 21, 505 (1961); 22, 598 (1962);

FIGURE CAPTION

Relation between the $I = 0$ $\pi\pi$ $S$ wave scattering length $a_0$ and the antibound state masses. $m_0$ is the $I = 0$ state mass, $m_2$ is the $I = 2$ state mass.