NUMERICAL COMPUTATION OF FIELD DISTRIBUTION AND FREQUENCY
IN THE LOWER PASSBANDS OF A SYMMETRICAL PERIODIC STRUCTURE

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ABSTRACT

In the past, relaxation methods have been used to compute the electromagnetic fields and the frequency of a periodic structure having a symmetry plane and rotational symmetry about its axis, at modes which are multiples of \( n \). These computations involve determining one scalar function over a half-cell of the structure.

In the present paper, it is shown that by determining two scalar functions over a half-cell of the structure, it is possible to compute by relaxation methods the electromagnetic fields and the frequency at any mode in the lower passbands of a symmetrical periodic structure, for a travelling wave as well as for a standing wave. All quantities of interest: shunt impedance, transit time factor, quality factor \( Q \), group velocity, may then be computed with good accuracy for any operating mode of the structure.

Numerical results are given for the five lowest passbands of an Alvarez structure without stems, originally designed to accelerate protons of 18.737 MeV in its lowest 0 mode.

I. Theory

Partial differential equation and boundary conditions

Consider a periodic structure with rotational symmetry about its z-axis. If the electromagnetic fields are also assumed to have rotational symmetry about the z-axis (i.e., \( \frac{d}{dz} = 0 \)), Maxwell's equations split up into two groups in the system of cylindrical coordinates \( z, r, \phi \). One group contains only the field components \( E_z, E_r, H_\phi \) (E-type of field), whereas the other group contains only the field components \( H_z, H_r, E_\phi \) (H-type of field). The computation of the electromagnetic fields will be presented here for an E-type of field, although the same procedure applies as well for
an $H$-type of field.

In the case of an $E$-type of field, the field components are related by the equations

\[ j\omega \varepsilon_o \frac{\partial E_r}{\partial z} = \frac{3H_x}{3r} \]
\[ j\omega \varepsilon_o \frac{\partial E_z}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} \left( rH_\phi \right) \]
\[ \frac{\partial E_r}{\partial z} + \frac{\partial E_z}{\partial r} = -j\omega \mu_o H_r \]

Introducing the scalar function

\[ j\nu = rH_\phi \]

casts the first two equations (1) into the form

\[ \omega \varepsilon_o E_r = -\frac{1}{r} \frac{\partial \nu}{\partial z} \]
\[ \omega \varepsilon_o E_z = \frac{1}{r} \frac{\partial \nu}{\partial r} \]

while the last equation (1) may be rewritten as

\[ \frac{\partial^2 \nu}{\partial r^2} - \frac{1}{r} \frac{\partial \nu}{\partial r} + \frac{\partial^2 \nu}{\partial z^2} + k^2 \nu = 0 \]

where \[ k^2 = \omega \varepsilon_o \mu_o \]

From (3) it is easy to show that the curves $\text{Re}(Ve^{j\omega t}) = \text{constant}$ at a given time in a $(r,z)$ plane represent the electric field lines at that time. Since they are normal to lossless metallic boundaries, one must have $\frac{\partial}{\partial n} \text{Re}(Ve^{j\omega t}) = \text{Re}(\frac{\partial}{\partial n} e^{j\omega t}) = 0$ on metallic boundaries, for all values of $t$. This implies the condition

\[ \frac{\partial \nu}{\partial n} = 0 \]

on metallic walls.

On the axis $r = 0$, with (2) and (3) we have the boundary conditions

\[ \nu = 0 \]
\[ \frac{\partial \nu}{\partial r} = 0 \]

The equations (4) to (7) apply to any type of $V$, standing or travelling wave.

From now on, we assume that $V$ represents a travelling wave having a phase shift $\beta_o L$ per cell in the positive $z$-direction (where $L$ is the length of a geometrical period of the structure). Then

\[ V(r,z + L) = e^{-j\beta_o L} V(r,z) \quad \text{or} \quad V(r,z - L) = e^{j\beta_o L} V(r,z) \]

Therefore $V$ is complex when $\beta_o L$ is not a multiple of $\pi$. Since losses are neglected, the differential equation (4) and the boundary conditions (5) to (7) for $V$ are real, so that $[V(r,z)]^*$ also represents a possible wave at the same frequency. From the complex conjugate of (8), this wave experiences a phase-shift $\beta_o L$ per cell in the negative $z$-direction.
Putting

$$V(r,z) = \phi_1(r,z) - j \phi_2(r,z)$$  \(9\)

where \(\phi_1\) and \(\phi_2\) are real, it follows that \(\phi_1\) and \(\phi_2\) are real solutions of Maxwell's equations at the same frequency. Their electric field lines, which are given by the curves \(\phi_1 = \text{constant}\) or \(\phi_2 = \text{constant}\), do not move with time: \(\phi_1\) and \(\phi_2\) thus represent pure standing waves.

From (8), the travelling wave \(V\) is known everywhere once it is known for \(-\frac{L}{2} \leq z \leq \frac{L}{2}\). In order to determine \(V\) completely in the cell \(-\frac{L}{2} \leq z \leq \frac{L}{2}\), we have the equations (4) to (7); but we still need boundary conditions at the planes \(z = -\frac{L}{2}\) and \(z = \frac{L}{2}\). Such boundary conditions are provided by equation (8). Indeed, this equation also implies that

$$\frac{3V}{3z} (r, z + L) = e^{-j8_0L} \frac{3V}{3z} (r, z)$$  \(10\)

and similar relations for all partial derivatives of \(V\) with respect to \(z\). In particular, for \(z = -\frac{L}{2}\) we have

$$V(r, \frac{L}{2}) = e^{-j8_0L} V(r, -\frac{L}{2})$$  \(11\)

and

$$\frac{3V}{3z} (r, \frac{L}{2}) = e^{-j8_0L} \frac{3V}{3z} (r, -\frac{L}{2})$$  \(12\)

Similar relations for all partial derivatives of \(V\) with respect to \(z\) then follow from (11), (12) and the differential equation (4). The boundary conditions (11), (12) combined with the equations (4) to (7) are thus sufficient to ensure that the relation (8) is satisfied for any \(z\). Therefore, the differential equation (4) combined with the boundary conditions (5) to (7) and (11), (12) determine \(V\) completely in the cell \(-\frac{L}{2} \leq z \leq \frac{L}{2}\). For the real functions \(\phi_1\) and \(\phi_2\), equations (4) to (7) keep the same form, whereas the boundary conditions (11), (12) become

$$\begin{cases} \phi_1\left(\frac{L}{2}\right) = \phi_1\left(-\frac{L}{2}\right) \cos \beta_0 L - \phi_2\left(-\frac{L}{2}\right) \sin \beta_0 L \\ \phi_2\left(\frac{L}{2}\right) = \phi_1\left(-\frac{L}{2}\right) \sin \beta_0 L + \phi_2\left(-\frac{L}{2}\right) \cos \beta_0 L \end{cases}$$  \(13\)

* The curves \(\phi_1 = \text{constant}\) or \(\phi_2 = \text{constant}\) also represent the electric field lines of the travelling wave \(V\) at times given by \(\omega t = 0\) or \(\omega t = \frac{n}{2}\) respectively.
and

\[
\begin{align*}
\frac{3\Phi_1}{3z} \left( \frac{L}{2} \right) &= \frac{3\Phi_1}{3z} \left( -\frac{L}{2} \right), \cos \beta_o L - \frac{3\Phi_2}{3z} \left( -\frac{L}{2} \right), \sin \beta_o L \\
\frac{3\Phi_2}{3z} \left( \frac{L}{2} \right) &= \frac{3\Phi_2}{3z} \left( -\frac{L}{2} \right), \sin \beta_o L + \frac{3\Phi_2}{3z} \left( -\frac{L}{2} \right), \cos \beta_o L
\end{align*}
\]  
(14)

The functions \( \Phi_1 \) and \( \Phi_2 \) are coupled together by \( \sin \beta_o L \). Inside a passband, \( \beta_o L \) is not a multiple of \( \pi \), so that none of these functions vanishes identically. On the other hand, at the edge of a passband, \( \beta_o L \) is a multiple of \( \pi \), and the coupling between \( \Phi_1 \) and \( \Phi_2 \) entirely disappears. The very special case of confluence of two passbands is the only case where two linearly independent standing waves \( \Phi_1 \) and \( \Phi_2 \) exist at the same frequency \( \beta_o \); in normal cases, there is only one real solution to Maxwell's equations in 0 or \( v \) mode (except for a constant factor).

Therefore, at \( n \pi \) modes (\( n \), integer), the electromagnetic fields in a periodic structure can be computed by determining in general one real function over a full cell of the structure; inside a passband or at a confluent \( n \pi \) mode, two real functions \( \Phi_1 \) and \( \Phi_2 \) must be determined over a full cell.

The eigenvalue \( k^2 \) may be obtained by equating the stored electric and magnetic energies of a travelling wave \( V \) in a full cell. Using (2) and (3) one gets

\[
k^2 = \frac{\int_{\text{full cell}} \left[ \frac{3V}{3r} \right]^2 + \left[ \frac{3V}{3z} \right]^2 \frac{dr}{r} \, dz}{\int_{\text{full cell}} |V|^2 \frac{dr}{r} \, dz}
\]  
(15)

It may be shown that this expression is stationary against small deviations of \( V \) from its correct value, as long as the boundary conditions (6) and (11) are still satisfied.

From the real part of the flux of the Poynting vector in the positive \( z \)-direction through any cross-section \( z = \) constant, the group velocity of the travelling wave along the \( z \)-axis is computed as

\[
\frac{V_c}{c} = \frac{kL}{k}
\]

\[
\operatorname{Im} \int_{\text{constant } z} V \frac{3V}{3z} \cdot \frac{dr}{r} = kL \int_{\text{constant } z} \frac{V \frac{3V^*}{3z} \frac{dr}{r}}{k}
\]

\[
\int_{\text{full cell}} |V|^2 \frac{dr}{r} \, dz = \int_{\text{full cell}} \left[ \frac{3V}{3r} \right]^2 + \left[ \frac{3V}{3z} \right]^2 \frac{dr}{r} \, dz
\]

(16)

The expressions (15) and (16) yield a point of the dispersion curve and its slope, for any mode \( \beta_o L \).
Case of a structure having a symmetry plane at \( z = 0 \)

In this case, \( V(r,-z) \) is also a solution of Maxwell's equations at the same frequency. From (8), it represents a travelling wave which experiences a phase-shift \( \beta L \) per cell in the negative \( z \)-direction. Therefore, it must be identical to \( \left[ V(r,z) \right]^* \) except for some constant factor, which may be taken as unity.\(^1\) With such a choice we have

\[
V(r,-z) = \left[ V(r,z) \right]^* \tag{17}
\]

and from (9):

\[
\phi_1(r,-z) = \phi_1(r,z) \quad \phi_2(r,-z) = -\phi_2(r,z) \tag{18}
\]

The functions \( \phi_1 \) and \( \phi_2 \) are then respectively even and odd in \( z \); in the following we shall denote them as \( \phi_S \) (symmetrical) and \( \phi_A \) (antisymmetrical) with respect to \( z = 0 \).

Because of (17), the travelling wave

\[
V(r,z) = \phi_S - j\phi_A
\]

is known in the full cell \(-\frac{L}{2} \leq z \leq \frac{L}{2}\) once it is known in the half-cell \( 0 \leq z \leq \frac{L}{2} \).

In order to obtain the boundary conditions for \( \phi_S, \phi_A \) at the planes \( z = 0 \) and \( z = \frac{L}{2} \), we notice from (17) that

\[
-\frac{3V}{\partial z}(r,-z) = \left[ \frac{3V}{\partial z}(r,z) \right]^* \tag{19}
\]

When (17) and (19) are introduced with \( z = \frac{L}{2} \) into (11) and (12), there results the conditions

\[
\text{Im}\left[ e^{rac{j\beta_0 L}{2}} V(r, \frac{L}{2}) \right] = 0 \quad \text{and} \quad \text{Re}\left[ e^{rac{j\beta_0 L}{2}} \frac{3V}{\partial z}(r, \frac{L}{2}) \right] = 0 \tag{20}
\]

which may be rewritten as

\[
\phi_S \sin \frac{\beta L}{2} - \phi_A \cos \frac{\beta L}{2} = 0 \quad \text{and} \quad \frac{\partial \phi_S}{\partial z} \cos \frac{\beta L}{2} + \frac{\partial \phi_A}{\partial z} \sin \frac{\beta L}{2} = 0
\]

at \( z = \frac{L}{2} \) \tag{21}

whereas from (18)

\[
\phi_A = 0 \quad \text{and} \quad \frac{\partial \phi_S}{\partial z} = 0 \text{ at } z = 0 \tag{22}
\]

In the half-cell \( 0 \leq z \leq \frac{L}{2} \), the functions \( \phi_S \) and \( \phi_A \) are completely determined by the differential equation (4) combined with the boundary conditions (5) to (7) and (21),(22).
They are coupled together by the conditions (21), as long as \( \sin \beta_0 L \neq 0 \). Therefore, in a periodic structure having a symmetry plane, the electromagnetic fields can be computed at any mode by determining two real functions \( \phi_S \) and \( \phi_A \) over a half-cell of the structure; at \( n \pi \) modes, the standing waves \( \phi_S \) and \( \phi_A \) are entirely independent of each other, and correspond in general to different frequencies (except in the case of confluence).

The frequency and the group velocity are always given by (15) and (16); but now, by symmetry considerations, the volume integrals may be computed over a half-cell only. It may be shown \(^1\) that the frequency expression (15) is stationary against small deviations of \( \phi_S \) and \( \phi_A \) from their correct values, provided the boundary conditions (6) and those conditions in (21), (22) which involve only the functions themselves (not their normal derivations), are still satisfied.

**Numerical computation by over-relaxation**

Using the well-known five-point method\(^4\), the differential equation (4) is replaced by a finite-difference equation where the values of \( \phi_S \) and \( \phi_A \) are taken at the nodes of a grid of orthogonal coordinate lines with a square mesh. At irregular nodes (nodes which are so close to a boundary, that one or two of the neighbouring nodes are missing), the boundary conditions are used to compensate for the missing neighbour(s). Near \( z = \frac{L}{2} \), the boundary conditions (21) couple \( \phi_S \) and \( \phi_A \) together, so that the finite-difference expression for \( \frac{\partial^2 \phi_S}{\partial z^2} \) involves \( \phi_A \) and vice-versa. Therefore, the successive iterations for \( \phi_S \) and \( \phi_A \) must be made alternately. An over-relaxation factor \( \theta \) is used to speed up convergence; its value is determined by trial as the largest value beyond which convergence disappears altogether. At the end of the relaxation, all quantities of interest: shunt impedance, transit time factor, quality factor \( Q \), can be calculated by numerical integrations.

**Initial loading.** The initial values of \( \phi_S \) and \( \phi_A \) which are used in the first iteration cycle are of paramount importance for a fast convergence of the relaxation process towards the wanted mode. The method of computation hitherto described has been applied to an Alvarez structure without stems, with the symmetry plane \( z = 0 \) taken in the middle of a gap (see figure 1). In order to get a rather good approximation of \( \phi_S \) and \( \phi_A \) for the initial loading, the actual drift tubes were replaced by drift tubes having the same diameter, \( 2a \), and the same length, but with square corners and no hole. This simplified problem was then solved following the method used by Walkinshaw and Bell\(^5\) for a disc-loaded waveguide with sharp edged discs. The principle of the method is that the fields of a travelling wave having a phase shift \( \beta_0 L \) per cell can be completely determined (as Fourier series in \( z \) involving Bessel functions in \( r \)) in both the inner region \( 0 < r < a \) and the outer region \( r > a \), if the frequency is known.
and either $E_z$ or $H_\phi$ is known at the radius $r = a$. Since neither of these conditions is met, the field $E_z$ (or $H_\phi$) at $r = a$ is represented as a linear combination of conveniently chosen functions $E_n$ which do not depend on the frequency, so that each field component can be computed, at a given frequency, as a linear combination of known functions. If the condition

$$\int_{z = \frac{L}{2}}^{z = -\frac{L}{2}} \left[ \frac{E_n}{r} \times (H_{\text{outer}}^* - H_{\text{inner}}^*) \right]_{r = a} \cdot \overrightarrow{ds} = 0$$

where $\overrightarrow{ds}$ is normal to the cylinder of radius $r = a$

is imposed for each function $E_n$, there results a determinantal equation which determines the frequency; it can be shown that this value of the frequency is stationary against small deviations of the field $E_z$ at $r = a$ from its correct value. This approximation of the frequency and subsequently of the fields, will be the better, the closer the trial $E_z$ is to its correct shape. Since from (3), the real part of $E_z$ is symmetrical with respect to $z = 0$, as that of $V$, and its imaginary part is antisymmetrical, we choose a two-component function for $E_z$ at $r = a$, an even and an odd one, by taking

$$E_z = C \frac{1}{\sqrt{1 - \left(\frac{2a}{\tilde{g}}\right)^2}} - jD \frac{\sin \frac{2\tilde{g}z}{\tilde{g}}}{\sqrt{1 - \left(\frac{2a}{\tilde{g}}\right)^2}} \quad \text{for} \quad 0 < |z| < \frac{\tilde{g}}{2}$$

(23)

where $C$ and $D$ are real parameters, and $E_z = 0$ for $\frac{\tilde{g}}{2} < |z| < \frac{L}{2}$.

The region in between the hypothetical square corners and the actual rounded corners of the drift tube, as well as the bore, were loaded with a zero potential.

**Intermediate loading.** Following the procedure described by Martini and Warner, the values of $H_\phi$ at $r = a$ were used, together with the most recent estimate (15) of the frequency, to compute analytically the potential $V$ in the region $r > 2a$. These analytical values of the potential were then considered as frozen, and a number (typically 12) of iterations were limited to the central part of the cell, up to a grid line near $r = 2a$. New estimates (15) of the frequency was then computed, and the new values of $H_\phi$ at $r = a$ were used to start a new cycle. By limiting the iterations to the central part of the cell, this procedure reduces the computation time substantially.

**II. Results**

The above method has been applied to compute the five lowest passbands and the field distribution in an Alvarez structure without stems, having the following dimensions (see figure 1):

$$2b = 150.0 \text{ mm} \quad L = 47.1944 \text{ mm} \quad \rho_2 = 6.164 \text{ mm}$$

\*This line is taken as a Dirichlet boundary line for the central part of the cell.
These dimensions are the arithmetic mean of the 41 cells of a scaled model which was built in 1966 for the second tank of the CERN proton linac, with a reduction factor 1 : 6.181; they correspond to a proton energy of 18.737 MeV in the lowest 0 mode. Figure 2 shows the computed dispersion curves of a uniform Alvarez structure having this cell as geometrical period. The dispersion curves for the \( E_0 \) modes of the empty guide are also drawn on the figure; space-harmonics have been used in order to restrict the phase-shift \( \phi_0 L \) per cell to the interval \((0, \pi)\). It is seen that the dispersion curves of the drift tube-loaded structure follow closely those of the empty guide, except that they never cross each other; instead, they shear off from one another, because of the coupling between modes which is introduced by the drift tubes. In 1967, measurements were made at CERN on the scale model of tank 2, with expanded polystyrene as dielectric, so that it was possible to suppress the stems supporting the drift tubes. The modes of the experimental structure with variable cell length were identified by counting the number of zeros of \( E_z \) on the axis, as measured by a bead perturbation technique. The first passband could be measured up to the 33rd mode (for which \( \phi_0 L = \frac{33}{44\pi} \)); it can be seen on figure 2 that this is precisely the highest mode of this passband which does not interfere with the second passband. The higher modes of the first passband are scattered among the lower modes of the second passband, and therefore it is difficult to identify them experimentally. On the other hand, the modes of the second passband have been identified up to the 9th. Taking into account the relative permittivity of the expanded polystyrene (measured value: about 1.026), and with the above definition of the modes for the variable cell structure, the measured resonant frequencies of this structure coincide with the computed resonant frequencies of the uniform average structure to a few parts in a thousand; the difference tends to increase near the \( \pi \) mode of the first passband, where the longer cells of the variable structure are progressively cut off.

The electric field lines of the standing waves \( \phi_s \) and \( \phi_A \) are shown in figures 3 to 6, for four typical modes in the three lowest passbands. It is seen how these modes are obtained by deformation of the symmetrical and antisymmetrical standing waves produced by the corresponding \( E_0 \) modes of the empty guide, which can be found in figure 2. The numerical value which is indicated near an electric field line is the constant value which \( \phi_s \) (or \( \phi_A \)) assumes along that line; the direction of the electric field is deduced at once from equations (3).

Figures 3 to 6 also show the variation of the amplitude \(|E_z|\) and phase \( \psi \) of the travelling wave electric field \( E_z = |E_z| e^{-j\psi} \) along the z-axis. The amplitude is normalised in such a way that

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*The latter modes occupy the same frequency range as the higher modes of the first passband, but they are closer to each other in frequency.*
From (3), (6) and (7), the phase $\psi$ of $E_z$ on the z-axis is the same as the phase of the travelling wave $V$. Because of (17), $E_z$ is real at $z = 0$; therefore, $\psi = 0$ modulo $\pi$ at $z = 0$. The value of $\psi$ is chosen in such a way that inside a half-cell $0 < z < \frac{L}{2}$, the phase varies continuously with the operating mode. On the other hand, at $z = \frac{L}{2}$, the first equation (20) implies that $\psi = \frac{\pi}{2} L$ modulo $\pi$. From the figures 3 to 6, it is seen that in the bore, the phase is practically a constant from the edge of the gap up to near the centre of the drift tube; the reason being that in this region, the fields reduce essentially to a single evanescent wave. Near the centre of the drift tube, another evanescent wave coming from the adjacent gap mixes up with the first one, thereby producing a rapid change of the phase.

Although the process of successive overrelaxation might diverge for the higher passbands, in practice even during several hundreds of iterations, there seemed to be no convergence problem with the initial loading which has been described above, as long as the relaxation factor is not taken too large. With 8798 mesh points in a half-cell, a computation time of 8 minutes on the CDC 6600 is generally sufficient to reduce the relative change in frequency to a few parts in $10^5$ between 12 successive iterations. However, the discretization error is likely to be much larger than this figure; first estimates, which need further check in the future, indicate that the overall accuracy should be a few parts in $10^3$ for the frequency, and a few units in the 2nd decimal place for the ratio $\sqrt{g/c}$.

Figure 1. Unit cell of an Alvarez structure without stems.
Figure 2

First five dispersion curves of a uniform Alvarez structure corresponding to the average cell of tank 2 model (18.737 MeV protons at the lowest 0 mode).

- dispersion curves of the empty guide for $E_0$ modes.
- dispersion curves of the guide loaded with drift tubes. The dots indicate the points which have been computed by relaxation.
Figure 3
1st branch, $\beta L = \frac{29}{41}$ $\pi$, 3235 MHz

Figure 4
2nd branch, $\beta L = \frac{32}{41} \pi$, 3764 MHz

Electric field lines of $\phi_S$ and $\phi_A$. Amplitude and phase of the travelling wave electric field along the axis.
Figure 5
2nd Branch, $\frac{\omega L}{c} = \frac{37}{41}$, 3673 MHz

Figure 6
3rd Branch, $\frac{\omega L}{c} = \frac{33}{41}$, 4047 MHz

Electric field lines of $\phi_s$ and $\phi_A$. Amplitude and phase of the travelling wave electric field along the axis.
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Distribution (open)
Abstract sent to ISR, MPS and SI Scientific Staff