On the factorization method in quantum mechanics

J. OSCAR ROSAS-ORTIZ

Departamento de Física Teórica, Universidad de Valladolid
E-47011 Valladolid, Spain
and
Departamento de Física, CINVESTAV-IPN, A.P. 14-740
07000 México D.F., Mexico.

Abstract

New exactly solvable problems have already been studied by using a modification of the factorization method introduced by Mielnik. We review this method and its connection with the traditional factorization method. The survey includes the discussion on a generalization of the factorization energies used in the traditional Infeld and Hull method.

1 Introduction

Since the beginning of the Quantum Mechanics (QM) there exists only a narrow set of exactly solvable physical problems. This set includes the well known harmonic oscillator and hydrogen-like potentials, the Morse potential, the square well potential and a few others. The study of such problems has been traditionally done by using the method of orthogonal polynomials and the factorization method. In principle, it is no matter the method used in the construction of the analytical solutions to the Schrödinger equation, however, the factorization (introduced by Schrödinger [1] and Dirac [2]) avoids the use of cumbersome mathematical tools and it has been successfully applied as an elegant approach in solving essentially any problem for which there exists exact solution. This is clear from the classification and study of the factorizable second order differential equations given by Infeld and Hull [3]. It is remarkable that, although the factorization method was developed to solve the eigenvalue problem related with the time-independent Schrödinger equation, it is also a very powerful tool in the study of the recurrence relations obeyed by special functions [4].
Until the beginning of the 1980’s decade, it was a very common opinion that the method was completely explored. The change of mentality became after the seminal work of Mielnik [5]. The fundamental idea introduced by Mielnik in such paper is to consider not the particular but the general solution to the Riccati type equation connected with the Schrödinger-Dirac-Infeld-Hull (SDIH) approach. Hence, the Mielnik’s method takes into account an important term of the factorization operators unnoticed by the traditional SDIH method. The first application of this modified method was developed in the study of the harmonic oscillator by Mielnik himself, giving rise to the construction of a one-parameter family of new exactly solvable potentials, different than the harmonic oscillator potential but having the same spectrum as it. Early on, the Mielnik method was applied by Fernández in the construction of a one-parameter family of new exactly solvable radial potentials, isospectral to the hydrogen-like radial one [6]. In the same year Nieto gave the links between this modified SDIH method and the supersymmetric (SUSY) approach to QM [7]. Its connection with Darboux transformation [8] has been discussed by Bagrov and Samsonov [9] (see also [10]), and some links with the inverse scattering method have been given by Zakhariev et al. [11]. By now the Mielnik method has been maturing for one and a half decade, and far from becoming a closed chapter in solving the Schrödinger equation, it is continuously reviewed and applied in diverse prospects [12-16]. For more historical links and related approaches see the interesting account given by Rosu in this volume.

In this contribution we shall present a general survey of the factorization method. Our main interest is to state clearly the differences between the SDIH and the Mielnik methods and, at the same time, to give the introductory material allowing their application in the derivation and study of new exactly solvable problems. In Section 2 we will review the SDIH method. Section 3 deals with the Mielnik method. Finally, in Section 4, we shall review the further generalizations of the Mielnik’s approach recently introduced in [14, 15]. Each section contains the corresponding harmonic oscillator and hydrogen-like potentials as particular cases.

2 Traditional factorization

The Schrödinger equation for a time-independent one-dimensional system is given by the usual eigenvalue equation

\[
H \psi(x) = \left[ -\left( \frac{\hbar^2}{2m} \right) \frac{d^2}{dx^2} + v(x) \right] \psi(x) = E \psi(x),
\]

where the potential \( v(x) \) can be a singular (e.g. the hydrogen-like potential) as well as a nonsingular (like the free particle or the harmonic oscillator potentials) real function, and the wave function \( \psi(x) \) must be square-integrable, that is, the integral \( \int |\psi(x)|^2 dx \) is finite. One often uses wave functions which are normalized, such that:

\[
\int |\psi(x)|^2 dx = 1,
\]
equation (2) is usually called the normalization condition.

As is well known, any physical problem must state not only the differential equation which is to be solved but also the boundary conditions which the solution must satisfy. Moreover, it is remarkable to notice that the determination of solutions satisfying of the boundary conditions is often as difficult a task as the solving of the differential equation. At the present state, it is apparent the essential role played by the normalization condition (2) in quantum theory because it represents the strongest boundary condition to be satisfied by the solutions to (1).

The spirit of the factorization method, as introduced by Schrödinger and Dirac, is to write the second order differential operator $H$ in (1) as the product of two first order differential operators $a$ and $a^\dagger$, plus a constant $\epsilon$. The form of such operators depends explicitly on the particular potential $v(x)$ we are dealing with, and implicitly on the factorization energy $\epsilon$, as we shall see below. We are going to sketch now a brief survey of the SDIH method. In order to eliminate a cumbersome notation let us rewrite the Schrödinger equation (1) in the dimensionless form

$$\left[ -\frac{d^2}{dx^2} + V(x) \right] \psi(x) = E\psi(x),$$

(3)

where $V(x) \equiv 2mv(x)/\hbar^2$ and $E \equiv 2mE/\hbar^2$. The factorization method aims to find two operators

$$a = \frac{d}{dx} + \beta_p(x), \quad a^\dagger = -\frac{d}{dx} + \beta_p(x),$$

(4)

such that the Hamiltonian

$$H = -\frac{d^2}{dx^2} + V(x)$$

(5)

can be written as

$$H = a^\dagger a + \epsilon.$$  

(6)

It is a matter of substitution to show that $\beta_p(x)$ has to satisfy the following Riccati equation:

$$-\beta_p'(x) + \beta_p^2(x) = V(x) - \epsilon.$$  

(7)

The SDIH method takes into account only the particular solution to the Riccati equation (7), this is the reason why we have labeled the $\beta$-functions, from the very beginning, with the subindex "$p". In the following table we reproduce the Infeld-Hull results for the harmonic oscillator as well as for the hydrogen-like potentials. The traditional representation for the harmonic oscillator is recovered from (4) and (6) by take $H \rightarrow H/2$, $a \rightarrow (1/\sqrt{2})a$, and $a^\dagger \rightarrow (1/\sqrt{2})a^\dagger$. 

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**Table:**

<table>
<thead>
<tr>
<th>Potential</th>
<th>Factorization Energy</th>
<th>Hamiltonian</th>
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<tbody>
<tr>
<td>Harmonic Oscillator</td>
<td>$\epsilon$</td>
<td>$H = -\frac{d^2}{dx^2} + V(x)$</td>
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<tr>
<td>Hydrogen-like</td>
<td>$\epsilon$</td>
<td>$H = -\frac{d^2}{dx^2} + V(x)$</td>
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<tr>
<td>Potential</td>
<td>Factorization energy</td>
<td>Particular solution to the Riccati’s equation</td>
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<tr>
<td>$V(x) = x^2$</td>
<td>$\epsilon = 1$</td>
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<td>“modified method” [3]</td>
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<tr>
<td>$V_l(r) - \frac{2}{r} + \frac{l(l+1)}{r^2}$</td>
<td>$\epsilon = -\frac{1}{r}$</td>
<td>$\beta_p(r, \epsilon) = l - \frac{1}{r}$ for a fixed $l$</td>
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Let us remark that, in general, given any $\epsilon \in \mathbb{R}$ for which there exists a solution to (7), there is a factorized expression for $H$ as it is given in (6). However, it is worthwhile noticing that, because the Riccati equation (7) is a non-linear differential equation, it is not, in general, integrable by quadratures. It therefore defines a family of transcendental functions [17], and the search of an adequate factorization energy permitting the construction of the $\beta_p$-functions is not an easy task. In fact, Infeld and Hull have reported only one factorization energy for each potential $V(x)$, hence, they have given the first step in the systematic construction of exactly solvable problems in Quantum Mechanics.

Now, let us take the product (6) in the opposite order and, by using (4) and (7), we get

$$aa^\dagger + \epsilon = H + 2\beta'_p(x).$$

(8)

In a first view, one might be tempted to believe that equation (8) brings out a new Hamiltonian $H + 2\beta'_p(x)$, but it is not always the case, as we are going to see in the next sections.

### 2.1 The harmonic oscillator revisited

As is well known, the solutions to (3), with $V(x) = x^2$, are given by Hermite polynomials type wave functions $\psi_n(x)$, whose corresponding eigenvalues are given by $E_n = 2n + 1$, $n = 0, 1, 2, \ldots$ The Infeld-Hull results are, in this case, $\epsilon = 1$ and $\beta_p(x) = x$, hence we have $a = d/dx + x$, and equation (8) can be rewritten as

$$H = -\frac{d^2}{dx^2} + x^2 = aa^\dagger - 1.$$

(9)

It must be clear that, in this case, equation (8) does not give any new Hamiltonian. Now, by using (9) and the corresponding factorization (6), we get the well known commutation rules satisfied by the operators $a$, $a^\dagger$ and $H$:

$$[a, a^\dagger] = 2, \quad [H, a] = -2a, \quad [H, a^\dagger] = 2a^\dagger.$$

(10)
Furthermore, it is well known that, for the harmonic oscillator problem, the operators of factorization $a^\dagger$ and $a$ are also the corresponding raising and lowering operators for the eigenfunctions of the Hamiltonian.

On the other hand, the method by itself gives us the chance of constructing a first solution to (3). In order to make clear this argument let us consider the equation (6), and suppose that there exists a function $\psi_N(x)$, such that $a\psi_N(x) = 0$, hence we will have $H\psi_N(x) = \epsilon\psi_N(x)$. The solution to this first order differential equation, providing the solution to (7) as given, is

$$\psi_N(x) \propto e^{-\int^x \beta_p(y) dy}.$$  

(11)

If $\psi_N(x)$ satisfies the normalization condition (2), then it is a wave function with eigenvalue $\epsilon$, else it is not a physical solution to the Schrödinger equation (3). For the system we are dealing with in this section, the function $\psi_N(x) = c_0 \exp(-x^2/2)$ is a physical eigenfunction of $H$ with the eigenvalue $\epsilon = E_N = 1$, hence $N = 0$, and $\psi_N(x) = \psi_0(x)$ is the ground state wave function of the harmonic oscillator.

Let us consider now the Hamiltonian (9). The solution to the first order differential equation $a^\dagger\psi_M(x) = 0$, given by $\psi_M(x) \propto \exp(x^2/2)$, is an eigenfunction of $H$ with eigenvalue $E_M = E_{-1} = -1 \not\in \{E_n\}$. It is clear that $\psi_M(x)$ is an unphysical solution because either it not satisfies the normalization condition (2), or because its eigenvalue $E_M$ is not allowed in the spectrum of $H$. We want to remark again that these unphysical solutions are usually discarded by the traditional methods of solution in Quantum Mechanics. In Section 4 we will have opportunity to show their relevance in the construction of new exactly solvable potentials.

### 2.2 The hydrogen-like potential revisited

The standard procedure to deal with hydrogen-like potentials in QM reduces to solve the eigenproblem for a particle in a one-dimensional effective potential $V_l(r) = l(l + 1)/r^2 - 2/r$, where $l = 0, 1, 2, ...$, is the azimuthal quantum number and $r$ is a dimensionless radial coordinate. By simplicity, instead of working with the standard radial wavefunctions $R_{l,n}(r)$, we will work with the functions $\psi_{n,l}(r) \equiv r R_{l,n}(r)$, with an inner product defined by $\langle \psi_{n,l}, \psi_{n,l}' \rangle \equiv 4\pi \int_0^{+\infty} \bar{\psi}_{n,l}(r)\psi_{n,l}'(r)dr < \infty$. As it is well known, the eigenvalues of the radial Hamiltonian

$$H_l = -\frac{d^2}{dr^2} + \frac{l(l + 1)}{r^2} = -\frac{d^2}{dr^2} + V_l(r),$$  

(12)

for a fixed $l$, are given by

$$E_{n,l} = E_{l,k} = -\frac{1}{(l+k)^2}; \quad k = 1, 2, 3, ...$$  

(13)

where $l + k = n$. Now, for the sake of clarity, let us take the following notation for the Infeld-Hull factorization operators of the Hamiltonian (12)

$$a_l = \frac{d}{dr} + \frac{l}{r} - \frac{1}{l}, \quad a_l^\dagger = -\frac{d}{dr} + \frac{l}{r} - \frac{1}{l},$$  

(14)
hence, the corresponding factorizations (6) and (8) can be written as
\[ a^*_l a_l - \frac{1}{l^2} = H_l, \quad a_l a^*_l - \frac{1}{l^2} = H_{l-1}. \] (15)
Notice that the right hand side equation (15) (e.g. (8)), can be rewritten as
\[ a_{l+1} a^*_l + 1 - \frac{1}{(l+1)^2} = H_l. \] (16)
In the present case, the operators of factorization \( a^*_l \) and \( a_l \), map the solutions of the corresponding equation (3), with a given energy, into solutions to (3) with the same energy but changing the value of the azimuthal quantum number \( l \). Hence, \( a^*_l \) and \( a_l \) are respectively the raising and lowering operators for the quantum number \( l \), besides we have \( \psi_{n,l+1}(r) \propto a_{l+1}^* \psi_{n,l}(r) \), and \( \psi_{n,l-1}(r) \propto a^*_l \psi_{n,l}(r) \).

Let us conclude this section with the observation that the physically relevant first solution to (3), derived from the SDIH method for the Hamiltonian (12), is given by \( \psi_M(r) \propto r^{l+1} \exp(-r/(l+1)) \), which is solution of \( a_{l+1}^* \psi_M(r) = 0 \). Hence, from (16) and (13), it is apparent that it is an eigenfunction of \( H_l \) with eigenvalue \( \epsilon = E_{l,1} = -1/(l+1)^2 \). On the other hand, the unphysical solution is now given by \( \psi_N(r) \propto r^{-l} \exp(r/l) \), with the forbidden eigenvalue \( E_{l,0} = -1/l^2 \).

3 Mielnik’s factorization

In this section we shall review the modification of the SDIH method as introduced by Mielnik. As we have discussed in the introduction, the Mielnik’s method aims to find two operators
\[ A = \frac{d}{dx} + \beta(x), \quad A^\dagger = -\frac{d}{dx} + \beta(x), \] (17)
factorizing the Hamiltonian (5) in the form
\[ H = A^\dagger A + \epsilon, \] (18)
where the function \( \beta(x) \) is the general solution of the Riccati equation
\[ -\beta'(x) + \beta^2(x) = V(x) - \epsilon. \] (19)
It is well known that, given any particular solution \( \beta_p(x) \) to (19), the corresponding general solution is given by two successive quadratures [17], and one can write
\[ \beta(x) = \beta_p(x) - \frac{d}{dx} \ln \left\{ \lambda - \int e^{\int \beta_p(y) dy} dx \right\}, \] (20)
where \( \lambda \) is an integration constant.
Now, by taking the factorization (18) in the opposite order we get
\[ AA^\dagger + \epsilon = H + 2\beta'_p(x) - 2 \frac{d^2}{dx^2} \ln \left\{ \lambda - \int e^{2 \int \beta_p(y) dy} dx \right\}. \] (21)
From Sections 2.1 and 2.2, it is clear that only the term $H + 2\beta_p(x)$, in the right hand side of (21), would correspond to the initial Hamiltonian $H$. The element of arbitrariness becomes then from the logarithmic term in such equation. Hence, we can define a new Hamiltonian $\tilde{H} = -(d^2/dx^2) + \tilde{V}(x)$, where

$$\tilde{V}(x) \equiv V(x) + 2\beta'(x)$$

(22)

and such that

$$\tilde{H} = AA^\dagger + \epsilon.$$  

(23)

Equations (22-23) open a new interesting problem: Is the following eigenvalue equation exactly solvable?

$$\tilde{H}\tilde{\psi}(x) = \tilde{E}\tilde{\psi}(x).$$  

(24)

The answer is affirmative and the corresponding solutions can be constructed by using the solutions of (3), with $\tilde{H}$ given in (18), and the following theorem:

**Intertwining Theorem**

Let $\psi(x)$ be an eigenfunction of $H$ with eigenvalue $E$, then $A\psi(x) \neq 0$ is an eigenfunction of $\tilde{H}$ with eigenvalue $\tilde{E}$. Similarly, if $\tilde{\phi}(x)$ is an eigenfunction of $\tilde{H}$ with eigenvalue $\tilde{E}$, then $A\tilde{\phi}(x) \neq 0$ is an eigenfunction of $H$ with eigenvalue $E$.

**Proof.**

$$\tilde{H}[A\psi(x)] = A[A^\dagger A + \epsilon]\psi(x) = E[A\psi(x)],$$

$$H[A\tilde{\phi}(x)] = A^\dagger[A^\dagger A + \epsilon]\tilde{\phi}(x) = \tilde{E}[A^\dagger A\tilde{\phi}(x)].$$  

Q.E.D.

Then we have $\tilde{\psi}(x) \propto A\psi(x)$, and $\psi(x) \propto A^\dagger \tilde{\phi}(x)$. Notice that, in the present approach, the operators of factorization $A^\dagger$ and $A$, are not the corresponding raising and lowering operators for the eigenfunctions of either $H$ or $\tilde{H}$. Now, as we have discussed in Section 2, if the solutions to (24) are wave functions, then they satisfy the normalization condition (2). Hence, we have

$$\langle \tilde{\psi}, \tilde{\phi} \rangle = \langle A\psi, A\psi' \rangle = \langle A^\dagger A\psi, \psi' \rangle = \langle (H - \epsilon)\psi, \psi' \rangle = (E - \epsilon)\langle \psi, \psi' \rangle$$

and, if $\psi(x) \in L^2(\mathbb{R})$, we can take $\tilde{\psi}(x) = (E - \epsilon)^{-1/2}A\psi(x)$. It is interesting to notice that, although we have used all the eigenfunctions of (18) in the construction of the new set $\{\tilde{\psi}(x) = (E - \epsilon)^{-1/2}A\psi(x) \mid \psi(x) \in L^2(\mathbb{R})\}$, it is not, in general, a complete set. Suppose there exists a function $\tilde{\psi}_{Me}(x)$ such that it is orthogonal to all $\tilde{\psi}(x)$, then we will have

$$\langle \tilde{\psi}_{Me}, \tilde{\psi} \rangle = (E - \epsilon)^{-1/2}\langle \tilde{\psi}_{Me}, A\psi \rangle = (E - \epsilon)^{-1/2}\langle A^\dagger \tilde{\psi}_{Me}, \psi \rangle = 0.$$  

Therefore, the solution to the first order differential equation $A^\dagger \tilde{\psi}_{Me}(x) = 0$, represents a possible missing element in the new basis $\{\tilde{\psi}(x)\}$. Such solution is given by

$$\tilde{\psi}_{Me}(x) \propto e^{\int^x \beta(y)dy},$$

(25)
and by using (20) we can rewrite it as

\[ \tilde{\psi}_{M\epsilon}(x) \propto e^{\int x \beta_p(y) dy} = \psi_M(x) \],

where \( \psi_M(x) \) is solution of \( A^\dagger \psi_M(x) = 0 \).

Now, if (25) satisfies the normalization condition (2), then it is a physical solution to the eigenvalue problem (24), with eigenvalue \( \epsilon \) (see equation (23)), and it has to be added to the new basis \( \{ \tilde{\psi}(x) \} \). In general, the eigenfunctions of \( \tilde{H} \) are given by \( \{ \tilde{\psi}_{M\epsilon}(x) \} \cup \{ \tilde{\psi}(x) \} \). Moreover, the Hamiltonian \( \tilde{H} \) is almost isospectral to \( H \), because its spectrum is the same as the spectrum of \( H \) plus a new level at \( \epsilon \): \( \{ \tilde{E} \} = \{ \epsilon, E \} \). In other words, we have solved the eigenvalue equation (24) by using the intertwining theorem and by considering the missing state (25).

Let us remark that the eigenproblem connected with \( H \) can be first solved by using the SDIH method and after by the Mielnik approach, which gives no new solutions for \( H \), but allows the construction of new exactly solvable Hamiltonians \( \tilde{H} \). Therefore, the Mielnik method is a further step in the systematic construction of exactly solvable problems in QM.

### 3.1 First order intertwining approach

As we have discussed in the last section, the Mielnik’s approach allows one to derive new Hamiltonians \( \tilde{H} \) departing from the factorization of a given Hamiltonian \( H \). It has been also shown that the eigenfunctions of \( H \) are an adequate point of departure in solving the eigenproblem connected with \( \tilde{H} \). Now, a natural question arises: Are the Hamiltonians \( H \) and \( \tilde{H} \) connected by a more direct relation? The answer is affirmative and it is clearly stated by the algebraic tool called the first order intertwining method [18]. Now, we are going to show how the basic results of the intertwining method can be recovered from the Mielnik’s approach. With this aim, let us apply the operator \( A \) to the right of equation (23), hence, we get

\[ \tilde{H} A = [AA^\dagger + \epsilon] A = A [A^\dagger A + \epsilon] = AH. \] (27)

Equation (27) means that the Hamiltonians \( H \) and \( \tilde{H} \) are connected by the operator \( A \). Notice that the multiplying order of the operators plays a fundamental role. Suppose now that \( \tilde{H} \) is a Hamiltonian with known solutions to the corresponding equation (3), then (27) immediately leads to the first part of the intertwining theorem. In the intertwining jargon it is said that \( A \) is an intertwiner operator because it intertwines (the eigenfunctions of) the operators \( H \) and \( \tilde{H} \). In this case the Hamiltonian \( \tilde{H} \) is called the initial one, while \( \tilde{H} \) is the intertwined Hamiltonian, which is to be determined by the method itself. On the other hand, the operator \( A^\dagger \) intertwines the Hamiltonians \( \tilde{H} \) and \( H \) in the form:

\[ HA^\dagger = [A^\dagger A + \epsilon] A^\dagger = A^\dagger [AA^\dagger + \epsilon] = A^\dagger \tilde{H}. \] (28)
In this case, considering the multiplying order of the operators, it is suitable to take \( \tilde{H} \) as the initial Hamiltonian, therefore, equation (28) allows one to recover the second part of the intertwining theorem. On the other hand, it is a matter of substitution to show that equations (19) and (22) are sufficient conditions validating the intertwining relationship (27). Moreover, all these equations lead, in a natural form, to the factorization of the initial Hamiltonian \( H \) with the multiplying order of (18). In a similar form, the sufficient conditions to satisfy (28) are given by

\[
\beta'(x) + \beta^2(x) = \tilde{V}(x) - \epsilon
\]

and

\[
V(x) = \tilde{V}(x) - 2\beta'(x),
\]

permitting the factorization of the Hamiltonian \( \tilde{H} \) as it has been made in (23).

In the last section we have done the assumptions validating the intertwining relationship (27). Now, we shall proceed in the opposite way, i.e., we will suppose that the eigenvalue problem (24) has been solved by the SDIH method and, therefore, we will focus on solving the corresponding equation for \( H \). With this aim, let us consider now the function \( \alpha(x) \) as a particular solution to (29), then the corresponding general solution is given by

\[
\beta(x) = \alpha(x) + \frac{d}{dx} \ln \left\{ \gamma + \int e^{-2 \int \alpha(y) dy} dx \right\},
\]

where \( \gamma \) is an integration constant (compare (31) with (20)). The corresponding eigenfunctions are given by those derived from the intertwining theorem, plus the missing state

\[
\psi_{N\epsilon}(x) \propto e^{-\int \beta(y) dy} = \frac{e^{-\int \alpha(y) dy}}{\gamma + \int e^{-2 \int \alpha(z) dz} dy}.
\]

Let us remark that, although it is no matter what kind of factorization (e.g. equations (18) or (23)) we use for the initial Hamiltonian in the intertwining approach (27-28), it is usually important to choose it adequately, in order to get a physically permissible intertwined Hamiltonian. Last statements will be clear in the next subsection, where we are going to apply the Mielnik method to the harmonic oscillator and to the hydrogen-like potentials by using the intertwining approach.

3.1.1 Classical isospectral potentials

**Harmonic oscillator**

We take the initial Hamiltonian \( H = -\frac{d^2}{dx^2} + x^2 + 2 \) as it has been factorized in (8-9). We want to factorize it in the form (23), hence, by making \( \tilde{V}(x) = x^2 + 2 \) in (29), we get the particular solution \( \alpha(x) = x \) (see the table of Section 2). Therefore, the corresponding intertwined potential (30) is given by

\[
V(x) = x^2 - 2 \frac{d}{dx} \left\{ \frac{e^{-x^2}}{\gamma + \int e^{-y^2} dy} \right\}.
\]
If $|\gamma| > \sqrt{\pi}/2$, the above potential has no singularity and behaves like $x^2$ for $x \to \pm \infty$; and so, one obtains here a one-parameter family of self-adjoint Hamiltonians $H$ in $L^2(\mathbb{R})$ [5]. The first eigenfunctions of $H$ are given by the intertwining theorem, while the corresponding missing state (32) is given by

$$\psi_{N\epsilon}(x) \propto e^{-x^2/2\gamma + \int_0^x e^{-y^2 dy}},$$

which is a square-integrable function because its behavior for $x \to \pm \infty$, provided that $|\gamma| > \sqrt{\pi}/2$.

**Hydrogen-like potential**

Let us take $H_l$ as it has been factorized in (15). We are going to factorize it in the form (18). By substituting $V_l(r)$ in (19) we get the particular solution $\beta\beta'(r) = l/r - 1/l$, with fixed $l$ (see the table in Section 2), and the corresponding intertwined potential $\tilde{V}_{l-1}(r)$ (see equation (22)) is given by

$$\tilde{V}_{l-1}(r) = -\frac{2}{r} + \frac{l(l+1)}{r^2} + 2 \frac{d}{dr} \left\{ \frac{r^{2l}e^{-2r/l}}{\lambda_l - \int_0^r y^{2l}e^{-2y/l}dy} \right\}, \quad l \geq 1.$$  (35)

If $\lambda_l > (2l)! \left( l/2 \right)^{2l+1}$, or $\lambda_l < 0$, for a fixed $l$, the third term has no singularities. Furthermore, for $r \to +\infty$, we have $\tilde{V}_{l-1}(r) = V_{l-1}(r)$, and therefore we obtain a one-parameter family of self-adjoint Hamiltonians $\tilde{H}_{l-1}$ in $L^2(\mathbb{R})$ [6]. The first eigenfunctions of $\tilde{H}_{l-1}$ are given by the intertwining theorem, while the corresponding missing state (26) is given by

$$\psi_{M\epsilon}(x) \propto \frac{r^l e^{-r/l}}{\lambda_l - \int_0^r y^{2l}e^{-2y/l}dy},$$

which is a square-integrable function because its behavior at $r = 0$, and in the limit $r \to +\infty$, provided that $\lambda_l > (2l)! \left( l/2 \right)^{2l+1}$, or $\lambda_l < 0$.

### 4 Further factorizations

In the last sections we have done a survey of the factorization methods available for the construction of analytical solutions to the Schrödinger equation. Now, we remark that, although the Mielnik method is a powerful tool in the derivation of new Hamiltonians whose corresponding eigenproblem is analytically solvable, not all the particular solutions to the Riccati equation reported in [3] have been taken into account for their generalization through the Mielnik’s approach. On the other hand, it is a matter of fact that the Infeld-Hull results in the table of Section 2 (e.g. [3]) correspond to only one factorization energy $\epsilon$ for each potential. Therefore, there is only one SDIH factorization available for the corresponding Hamiltonian and we have at hand only one
family of isospectral potentials connected with it. In the present section we are going to sketch some steps allowing the selection of new factorization energies. With this aim, we shall take $H$ in (18) as the initial Hamiltonian [19]. The transformation

$$\beta(x) = -\frac{d}{dx} \ln u(x), \quad (37)$$

leads (19) to the second order differential equation [15]

$$\left[ -\frac{d^2}{dx^2} + V(x) \right] u(x) = \epsilon u(x). \quad (38)$$

Hence, the $\beta$-functions are directly connected with the eigenfunctions of the initial Hamiltonian $H$, corresponding to the eigenvalue $\epsilon$. Notice that, if the factorization energy $\epsilon$ belongs to the spectrum $\{E\}$, then the solutions $u(x)$ to (38) are the physically relevant functions $\psi(x)$. Now, from Section 3.1.1, we know that the factorization energy $\epsilon = 1$, for the harmonic oscillator $H + 2$, does not belong to $E_{n+1}$. On the other hand, for the hydrogen-like potential we have $\epsilon = -1/l^2$, which is not allowed in $E_{l,K}$, $K > 0$. Hence, following [5] and [6], we shall consider just the cases where the factorization energy $\epsilon$ does not belong to the spectrum of $H$. Therefore, the solutions to (38) do not have direct physical meaning, but we have just seen that they naturally lead to the factorization of the Hamiltonians $H$ and $\tilde{H}$ in the spirit of Mielnik’s approach, providing as well the explicit form for the new potentials $\tilde{V}(x)$.

In order to solve (38) we take the factorization of $H$ as given in (6), hence we have two options:

(i) $a u(x) = 0$,

(ii) $a u(x) = w(x) \neq 0$; $a\dagger a u(x) = 0 \Rightarrow a\dagger w(x) = 0$.

Notice that $u(x) = \psi_N(x)$, with $\psi_N(x)$ given in (11), is the solution of (i), whereas $w(x) = \psi_M(x) = c_1/\psi_N(x)$, with $c_1$ a constant, is the solution to the right hand side equation (ii). Hence, the general solution to the left hand side equation (ii) is given by

$$u(x) = \psi_N(x) \left[ c_0 + c_1 \int \psi_N^2(y) dy \right]. \quad (39)$$

where $c_0$ is an integration constant. Now, by using (37) and (11), with $c_1 = -1$ and $c_0 = \lambda$, we get (20). The general solution of (19) is then constructed by the general solution of (38), with the unphysical eigenvalue $\epsilon$.

4.1 New isospectral potentials

In this section we are going to solve, explicitly, the eigenvalue problem (38) for the harmonic oscillator and the hydrogen-like potentials. Hence, our results will bring out new factorization energies permitting the construction of new one-parameter families of potentials isospectral to each one of the potentials mentioned before.
4.1.1 The hydrogen-like potential

In order to solve (38), with \( V(x) = V_l(r) \), let us write the factorization energy as
\[
\epsilon_l^{(k)} = -\frac{1}{(l + k)^2}, \quad k \neq K, \quad l > 0.
\]
It is clear that, for a fixed value of \( l \), we have \( \epsilon_l^{(k)} \neq E_{l,K}, \forall k \neq K \). Now we make the transformation \[15\]
\[
u_k = l r - 1 \frac{1}{(l + k)} - \frac{d}{dr} \ln \Phi_l^{(k)}(r), \quad l > 0.
\]
leading to a confluent hypergeometric equation for \( \Phi_l^{(k)}(r) \), whose general solution for the discrete values \( k = 0, -1, -2, ..., -(l - 1) \) is given by the linear combination of confluent hypergeometric functions \[20\]
\[
\Phi_l^{(k)}(r) = \frac{\Gamma(1 + |k|)}{\Gamma(2l + 2)} \frac{\Gamma(-2l)}{\Gamma(-2l + |k|)} \lambda_l^{(k)}, \quad l > 0.
\]
The general solution to the corresponding Riccati equation (19) arises after introducing (40) in (37), which gives:
\[
\beta_l^{(k)}(r) = \frac{l}{r} - \frac{1}{(l + k)} - \frac{d}{dr} \ln \Phi_l^{(k)}(r), \quad l > 0.
\]
The intertwined potentials \( \tilde{V}_l^{(k)}(r) \) have the same singularity at \( r = 0 \) as \( V_{l-1}(r) \), provided that \( \lambda_l^{(k)} \) takes only values in the domain \[15\]:
\[
\lambda_l^{(k)} \in \begin{cases} 
(-\infty, 1), & \text{for } |k| \text{ even}; \\
(1, \infty), & \text{for } |k| \text{ odd}.
\end{cases}
\]
By taking \( k = 0 \) and \( \lambda_0^{(0)} = 0 \), the last results reduce to the corresponding SDIH results \[3\], for the same value of \( k \) and \( \lambda_0^{(0)} = (2l)!(l/2)^{2l+1} \gamma_l^{-1} \), we get the Fernández results \[6\] and, finally, the corresponding Abraham and Moses results \[22\] arise for \( k = 0 \) and \( \lambda_0^{(0)} \rightarrow 1 \), with \( l = 1 \).

4.1.2 The harmonic oscillator

For the harmonic oscillator potential we take \( \tilde{V}(x) = x^2 \) as the initial potential, hence, in order to solve the corresponding equation (38), let us make the transformation
\[
\tilde{u}(x) = \Phi(x) \exp(-x^2/2),
\]
leading to a confluent hypergeometric equation for \( \Phi_l^{(k)}(r) \), whose general solution for the discrete values \( k = 0, -1, -2, ..., -(l - 1) \) is given by the linear combination of confluent hypergeometric functions \[20\]
\[
\Phi_l^{(k)}(r) = \frac{\Gamma(1 + |k|)}{\Gamma(2l + 2)} \frac{\Gamma(-2l)}{\Gamma(-2l + |k|)} \lambda_l^{(k)}, \quad l > 0.
\]
leading to a confluent hypergeometric equation for \( \Phi_l^{(k)}(r) \), whose general solution for the discrete values \( k = 0, -1, -2, ..., -(l - 1) \) is given by the linear combination of confluent hypergeometric functions \[20\]
\[
\Phi_l^{(k)}(r) = \frac{\Gamma(1 + |k|)}{\Gamma(2l + 2)} \frac{\Gamma(-2l)}{\Gamma(-2l + |k|)} \lambda_l^{(k)}, \quad l > 0.
\]
leading to a confluent hypergeometric equation for \( \Phi_l^{(k)}(r) \), whose general solution for the discrete values \( k = 0, -1, -2, ..., -(l - 1) \) is given by the linear combination of confluent hypergeometric functions \[20\]
\[
\Phi_l^{(k)}(r) = \frac{\Gamma(1 + |k|)}{\Gamma(2l + 2)} \frac{\Gamma(-2l)}{\Gamma(-2l + |k|)} \lambda_l^{(k)}, \quad l > 0.
\]
leading to a confluent hypergeometric equation for \( \Phi_l^{(k)}(r) \), whose general solution for the discrete values \( k = 0, -1, -2, ..., -(l - 1) \) is given by the linear combination of confluent hypergeometric functions \[20\]
\[
\Phi_l^{(k)}(r) = \frac{\Gamma(1 + |k|)}{\Gamma(2l + 2)} \frac{\Gamma(-2l)}{\Gamma(-2l + |k|)} \lambda_l^{(k)}, \quad l > 0.
\]
and by introducing instead of $x$ the variable $y = x^2$ we then find $\Phi(y)$ to satisfy a confluent hypergeometric type differential equation, whose general solution is given by [20]

$$\Phi(y) = _1F_1\left(\frac{1-\epsilon}{4}, \frac{1}{2}; y\right) + \nu \frac{\Gamma\left(\frac{3-\epsilon}{4}\right)}{\Gamma\left(\frac{1-\epsilon}{4}\right)} y^{1/2} _1F_1\left(\frac{3-\epsilon}{4}, \frac{3}{2}, y\right).$$

The general solution to the corresponding Riccati equation (29) arises after the transformation $\beta(x) = (d/dx) \ln \tilde{u}(x)$, which gives

$$\beta(x) = -x - \frac{d}{dx} \ln \Phi(x^2).$$

The intertwined potential becomes free of singularities if $\epsilon < 1$ and $|\nu| < 1$ [14, 21]. In particular, we can take $\epsilon(k) = -2k - 1$, with $k = 0, 1, 2, \ldots$ [14].

In this case, the SDIH results arise by taking $k = 0$ and $\nu = 0$, the Mielnik results [5] are recovered with $k = 0$ and $\nu^{(0)} = (\sqrt{\pi}/2)\gamma^{-1}$ and, by taking the same value of $k$, with $\nu \to 1$, the Abraham and Moses family is recovered. We conclude this Section remarking that the factorization energies $\epsilon^{(k)}_i$ and $\epsilon^{(k)}$, generalize the choice made for the SDIH and Mielnik factorizations of the hydrogen-like and harmonic oscillators potentials respectively.

### 4.2 Higher order factorizations

The Mielnik factorization allows us to construct further new families of exactly solvable potentials departing from any exactly solvable problem and iterating after the procedure. The new families will inherit the number of parameters from the old ones. The keystone of that construction is given by [14, 15, 16]:

$$\beta_{n+1}(x) = -\beta_n(x) - \left(\frac{\epsilon_{n+1} - \epsilon_n}{\beta_{n+1} - \beta_n}\right), \quad n = 1, 2, \ldots$$

where $n$ denotes the number of iterations.

Notice that the last equation works immediately for $\epsilon_{n+1} \neq \epsilon_n$. Hence, by giving $n$ different factorization energies one can iterate $n$ times the Mielnik’s approach constructing $n$th-parametric families of analytically solvable potentials. The results presented in this paper reproduce those recently derived in [14, 15]. Although we have shown how this method works by solving the discrete problems connected with two interesting physical systems, its applications include the study of systems with continuous spectrum (e.g. $\epsilon_{n+1} \to \epsilon_n$) [16].

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[16] B. Mielnik, L.M. Nieto and O. Rosas-Ortiz, The finite difference algorithm for higher order supersymmetry, preprint UVA.


[19] It is clear that the results and discussions presented in this section for $H$ will be valid for $\tilde{H}$ by taking the corresponding changes.

