THE STANDARD MODEL OF THE ELECTROWEAK INTERACTIONS
(except CP violation)
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1. THE BASIC STRUCTURE OF THE THEORY

1.1 Preliminaries

After having recalled that the first laboratory manifestations of the weak interactions go back to the end of the last century, with the works of Becquerel, Lord Rutherford and the Curies on radioactivity, these preliminary considerations about the theory of the weak interactions start with the work of Fermi, who introduced his celebrated theory of $\beta$ decay in 1934. This was three years after the invention by Pauli of the neutrino to explain the apparent non conservation of energy in $\beta$ transitions.

Today we write the Fermi interaction which accounts for the decay of the neutron as

$$L_F = \frac{G}{\sqrt{2}} \cos \theta_C \left( \bar{p} \gamma_\mu (1 + \alpha \gamma_5)n \right) \left( \bar{e} \gamma_\mu (1 + \gamma_5)v \right).$$  (1.1)

where $G$ is a constant with dimension of inverse mass squared, $\theta_C$ is the Cabibbo angle and $\alpha$ is the ratio of the axial to vector couplings of the nucleon. This interaction allows to calculate the neutron decay width as

$$\Gamma = \frac{G^2 \Delta^5}{60\pi} \cos^2 \theta_C (1 + 3\alpha^2) 0.47$$  (1.2)

where $\Delta = 1.29\text{MeV}$ is the neutron-proton mass difference and the last numerical factor in the right-hand-side would be 1 for a massless electron. Using the experimental values [1]

$$\tau_n = \frac{1}{\Gamma} = (887.0 \pm 2.0) s; \quad \alpha = -1.2573 \pm 0.0028$$  (1.3)

and, for the Cabibbo angle (to be defined more precisely later on), $\cos \theta_C = 0.97$, we obtain $G^{-1/2} = 250\text{GeV}$, setting in this way the typical scale for the weak interaction phenomena.

We now know that the interaction in Eq. (1.1) is mediated by the exchange of a $W$ boson. Indeed the exchange of a $W$ boson, at negligible momenta with respect to its mass $M_W$, gives rise to a 4-fermion interaction of the form in Eq. (1.1), if we assume an interaction of the $W$ itself

$$L_I = -igW^\mu_\mu J^+_\mu + h.c. \quad J^+_\mu = \frac{1}{\sqrt{2}} \left( \bar{p} \gamma_\mu (1 + \gamma_5)n + \bar{e} \gamma_\mu (1 + \gamma_5)v \right).$$  (1.4)
we identify \( \frac{g^2}{8M_W^2} \) with \( \frac{G}{\sqrt{2}} \), and we neglect, for the time being, the Cabibbo angle. As a matter of fact, in this way one does not only obtain the interaction in Eq. (1), but the full current-current interaction, which contains also purely hadronic as well as purely leptonic interactions. A major role in postulating the current-current form of the weak interaction Lagrangian was played by Feynman and Gell-Mann in 1957-58. Furthermore, by requiring that the dimensionless coupling \( g \) does not exceed, say, 1, one gets an upper bound on the W mass of about 110 GeV.

1.2 Towards a Gauge theory

Once we know that the interaction in Eq. (1.4) has to be incorporated into a gauge theory [2], the step that it takes to do so is not too long. We recall that a gauge field theory is minimally defined by the gauge group \( G \) and by the transformation properties under \( G \) of the matter multiplets, which can either have spin \( \frac{1}{2} \) or 0. For any generator of the gauge group \( T^a \), there is an associated vector boson \( V^a_L \). Furthermore, the transformation properties of the matter multiplets under \( G \) define how the generators are represented, by the matrices \( t^a \), when acting on them. Denoting by \( \Psi \) a column vector containing all the spin \( \frac{1}{2} \) matter fields, all taken with the same chirality, their interaction with the vector bosons is

\[
L_I = -ig V^a_L \bar{\Psi} \gamma_\mu t^a \Psi. \tag{1.5}
\]

Coming back to the special case of the weak interactions, two generators will have to be associated with the two charged W bosons: \( W^\pm \rightarrow T^\pm \). Our purpose is to find a minimum set of generators that form a closed algebra. To this end, we try to assume that no other fermion is involved in the weak interactions other than the ones entering into the interaction Lagrangian (1.4). It is useful to rewrite the current in Eq. (1.4) in the form

\[
J^+_\mu = \bar{N}_L \gamma_\mu \frac{\sigma^+}{2} N_L + \bar{L}_L \gamma_\mu \frac{\sigma^+}{2} L_L, \tag{1.6}
\]

where \( \sigma_i \) are the usual Pauli matrices, \( \sigma^\pm = \frac{1}{\sqrt{2}}(\sigma_1 \pm i\sigma_2) \), and we have organised the left handed fermion fields in doublets.
By comparison of Eq. (1.5) with Eq. (1.6) we see that the generators $r^{\pm}$ are represented by $a_{\pm}$. We are now in the position to calculate the commutator of $r^2$ and $r^1$ which gives

$$[r^+, r^-] = [\sigma^+, \sigma^-] = i \sigma_3 r^3 .$$

(1.8)

We have obtained in this way a closed algebra of generators, the one of SU(2), which includes other than the charged generators $T^{\pm}$, also a diagonal generator, which is therefore neutral. We do know in fact already another diagonal generator, the electric charge $Q$ itself, represented on the doublets $N_L$ and $L_L$ as $Q = T_3 + \frac{1}{2}$ and $Q = T_3 - \frac{1}{2}$ respectively. $Q$ therefore does not commute with $T^{\pm}$, as $T_3$ does not, but it can rather be written as a linear combination of $T_3$ itself and another neutral generator, the hypercharge $Y$, $Q = T_3 + Y$, which commutes with all the $T_i$. On both doublets, $Y$ is in fact proportional to the unit matrix, being $YN_L = \frac{1}{2} N_L$ and $Y_L = -\frac{1}{2} L_L$. The generators $\{T_i, Y\}$ form the closed algebra that we are looking for.

The most important result is that, in this process, we have been led to the embedding of electromagnetism into the overall scheme and, even more importantly, to the unavoidable introduction of one more neutral current generator, $Y$.

To be precise, electromagnetism is fully included only once we tell how the various generators also act on the right handed fermions. Phenomenologically, the charged current in Eq. (1.4) does not involve the right-handed fermions, for which we therefore require that they be annihilated by $T^{\pm}$ and so also by $T_3$. Furthermore, since we wish to maintain $Q = T_3 + Y$, the following hypercharges have to be assigned to the various right handed fields:

$$Y_{pR} = p_R, \quad Y_{nR} = 0, \quad Y_{eR} = -e_R, \quad Y_{vR} = 0.$$  

(1.9)

The picture is completed if we now replace the nucleons with the quarks, which is the correct description of hadrons at a more fundamental level. The left handed doublet $N_L$ is replaced by the left handed quark doublet

$$N_L = \frac{1+\gamma_5}{2} (p) ; \quad L_L = \frac{1+\gamma_5}{2} (e).$$

(1.7)
$Q_L = \frac{1 + Y_5}{2} \begin{pmatrix} u \\ d \end{pmatrix}, \quad (1.10)$

of hypercharge $YQ_L = \frac{1}{6} Q_L$, and the right handed nucleons by the right handed quarks, $u_R$ and $d_R$, of hypercharge $\frac{\gamma}{3}$ and $-\frac{1}{3}$ respectively.

1.3 The Gauge Lagrangian $L^g$

We have fully specified in this way the gauge Lagrangian of the unified weak and electromagnetic interactions. We are in fact in the position to write it down explicitly, which we are going to do in a compact but useful form.

The gauge group $G$ is SU(2)XU(1), with the SU(2) factor generated by the $T_i$ and the phase factor U(1) by the hypercharge $Y$. The crossed product in SU(2)XU(1) is there to recall that $T_i$ and $Y$ commute among each other. Let us call $W_\mu^i$ and $B_\mu$ the corresponding gauge bosons. Let us also organise the chiral fermions as a column vector $\Psi^T = (Q_L^T, u^c, d^c, L_L^c, e^c)$, with $T$ standing for transposed and $u^c, d^c, e^c$ for the charge-conjugated of the right handed fields. In this way, all the fermions in $\Psi$ are taken to be left handed, since the charge conjugation operation anticommutes with $\gamma_5$. One should, however, be careful in not confusing $u^c, d^c, e^c$ with the left handed $u_L, d_L, e_L$. The latter have charged current weak interactions whereas the former do not, as we have seen. Notice also that we have not introduced, among the components of $\Psi$, the right-handed neutrino, since it has both zero weak isospin and zero hypercharge. If we had introduced it, it would have had no gauge interactions but only a kinetic term.

The full minimal gauge Lagrangian for the SU(2)XU(1) gauge group can be written as

$L^g = -\frac{1}{4} \text{Tr} W_{\mu\nu}^i W_{\mu\nu}^j - \frac{1}{4} B_{\mu\nu} B_{\mu\nu} + i \bar{\Psi} D \Psi \quad (1.11)$

where \( W_\mu = W_\mu^i \frac{\sigma^i}{2} \), \( W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - ig \left[ W_\mu, W_\nu \right] \), \( B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \) and \( D_\mu = \partial_\mu - ig W_{\mu}^i \frac{\sigma^i}{2} - ig' YB_\mu \). As mentioned before, $t^i$ are the matrices representing the SU(2) generators $T^i$ on the fermion matter multiplet $\Psi$. The high reducibility under the gauge group $G$ of $\Psi$ makes the $t^i$ block diagonal. Notice that the factorized nature of the gauge group allows the introduction of two independent coupling constants, $g$ and $g'$, one for each factor of the gauge group itself.
The following is a customary and self-explanatory notation to indicate the transformation properties of the matter fermions under $SU(2) \times U(1)$:

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} = \left(2, \frac{1}{6}\right), \quad L_L = \begin{pmatrix} v_L \\ e_L \end{pmatrix} = \left(2, -\frac{1}{2}\right), \quad u^c = \left(1, -\frac{2}{3}\right), \quad d^c = \left(1, \frac{1}{3}\right), \quad e^c = (1,1).$$

One may have to recall that charge-conjugation changes the sign of all charges or, more generally, it sends a generator $t^a$ acting on a representation $R$ into $-t^{a*}$ when acting on the charge-conjugated representation.

1.4 Problems: Anomalies, Charge Quantization, ...

Before going further in the description of the theory, we want to pause for a while and comment on some aspects of it which are of great importance *per se* and may also very well indicate that what we have described so far is only a fragment of a more complete theory.

Up to now we have only been talking of a classical Lagrangian. It is clear however that we are interested in turning it into a quantum theory. The point now is that quantum field theories are in general plagued by "anomalies" [3], which may occur since the quantum field theory itself can be defined only as the limit of a theory with an ultraviolet cut-off, usually called $\Lambda$. If the regularised theory violates some symmetry present at the classical level, the symmetry itself may not be recovered when the cut-off is removed and even be completely lost in the full quantum theory. In turn, this would simply not be tolerable for the gauge symmetry itself, since it would undermine the very consistency of the theory. For a general gauge theory, on the other hand, it turns out that there is a simple criterion to assess the presence of an anomaly. In terms of the matrices $t^a$ representing the generators of the gauge group as acting on the matter fermions, all taken to be left handed as we did in the previous section for the weak interaction gauge group, the anomaly is completely controlled by the three-index symmetric tensor

$$D^{abc} = Tr\left(t^a t^b t^c\right)$$

(1.12)

with the trace going over all fermions. The absence of anomalies requires the vanishing of the tensor $D^{abc}$.

This consistency criterion has to be met, as a particular case, also by the Lagrangian of the electroweak interactions that we have described in the previous section. In principle, taking into account that there are 4 generators and the
symmetry under permutations of the indices, one would have to compute 10 quantities. In practice it is simple to verify that, for a generic SU(2)XU(1) theory with fermions occurring only in left-handed doublets or singlets, all anomalies are absent if and only if the sum of the electric charges of the doublet fermions, \( D = Tr_{doub}(Q) \), vanishes. Remarkably enough, as first noticed by Bouchiat, Iliopoulos and Meyer [4], one has, for the lepton and quark doublets respectively

\[
D_{\text{lept}} = -1 \quad \text{and} \quad D_{\text{quark}} = 3 \left( \frac{2}{3} - \frac{1}{3} \right) = +1
\]  

with the factor of 3 in the quark term coming from colour. The cancellation of the overall anomaly is the only existing bridge, in the Standard Model (SM), between quarks and leptons.

Another problem which may or may not be related to the anomaly cancellation just discussed is that of charge quantization. We refer to the phenomenological observation that the charge of the electron and the charge of the proton coincide with a relative precision, from tests at cosmological scales, at a relative level of about \( 10^{-21} \) [1]. This is in sharp contrast with the fact that, in the Lagrangian of the previous section, the electric charges of the various fermions are fixed by hand. Let us recall that \( Q = T_3 + Y \), namely the sum of a quantized (\( T_3 \)) and a non-quantized generator (\( Y \)). \( T_3 \) is quantized since it belongs to a non-abelian algebra of generators, whereas the abelian charge \( Y \) is not. As one can explicitly check, the factorized \( U(1) \) invariance of the Lagrangian (1.11) would not be disturbed if we had taken even relatively irrational hypercharges of the various representations for the fermions.

There may be a connection between the anomaly cancellation condition and the charge quantization condition that we have discussed. It is a fact that both conditions are naturally and automatically fulfilled by embedding the theory that we have outlined into a proper Grand Unified Theory [5] as, e.g., the one based on the \( SO(10) \) gauge group [6].

1.5 Global ("accidental") symmetries of \( L^G \)

The Lagrangian \( L^G \) written down in section 1.3 is far from being realistic. One thing for sure is that it has too many unwanted symmetries. These symmetries, other than the gauge symmetry itself, arise automatically in the minimal gauge Lagrangian. For this reason they have been generally called by Weinberg "accidental". This is an important point which deserves an explanation.
No other term could have been written in $L(\mathfrak{g})$ which is both gauge invariant and renormalizable. Technically, once gauge invariance is ensured, a term in a Lagrangian is renormalizable if it does have dimension in mass less than or equal to four. Other gauge invariant terms could have been added to the Lagrangian (1.11) but they would have necessarily been of dimension higher than four, namely non-renormalizable, and, as such, scaled by inverse powers of a characteristic mass, to be called $M$. The important point is that, although the Lagrangian of the electroweak interactions need not be renormalizable, all effects induced by the non-renormalizable operators would be suppressed, at energies $E$ lower than $M$, by powers of $\frac{E}{M}$. This would in particular also apply to the possible breaking phenomena of an "accidental" symmetry, since, by definition, an "accidental" symmetry is respected by all possible renormalizable terms, but it is not if non-renormalizable interactions are allowed in the Lagrangian. The bigger is $M$, the more are the chances that an "accidental" symmetry appears phenomenologically respected, even if it is violated in the Lagrangian.

After this digression, let us ask which are, if any, the "accidental" symmetries of $L(\mathfrak{g})$. To this end, let us first consider the free kinetic term of the matter fermions organised in $\Psi$, $L_{\text{kin}} = \bar{\Psi} \partial \Psi$. $\Psi$ consists of 15 left-handed Weil spinors, counting also the different colours for the quarks. As such, $L_{\text{kin}}$ has a overall $U(15)$ symmetry, since all these spinors can be unitarily transformed into each other without affecting the kinetic Lagrangian itself. What happens of this continuous symmetry after the gauging of the $SU(2) \times U(1)$ group? Recalling that $\Psi$ breaks down under $SU(2) \times U(1)$ into 5 different irreducible representations, the gauged Lagrangian of the fermion fields has the form

$$\begin{align*}
L_{\text{fermion}}(\mathfrak{g}) &= i \bar{\Psi} \mathcal{D} \Psi = i \sum_\alpha \bar{r}_\alpha D_\alpha r_\alpha,
\end{align*}$$

(1.14)

where $r_\alpha$ denotes any of the five irreducible representations and $D_\mu_\alpha$ is the covariant derivative acting on $r_\alpha$. It follows that $L_{\text{fermion}}(\mathfrak{g})$ is invariant, a part from the same gauge transformations, under five independent phase transformations acting separately on the $r_\alpha$. Actually, since a factor of the gauge group is itself a $U(1)$ phase transformation, generated by the hypercharge, only four independent $U(1)$ factors remain as ungauged global, as opposed to local or gauged, symmetries. They can be defined in such a way that the corresponding charges, according to Noether's theorem, are
\[ 3B = N(q) - N(q^c) = 3 \text{ (baryon number) }, \quad B_A = N(q) + N(q^c) \]
\[ L = N(l) - N(l^c) = \text{ lepton number }, \quad L_A = N(l) + N(l^c), \]
where \( N(q) \) (\( N(q^c) \)) and \( N(l) \) (\( N(l^c) \)) are the numbers of quarks (antiquarks) and
of leptons (antileptons) respectively. It is easy to see that it would be possible to
write down gauge invariant non-renormalizable 4-fermion operators that would
break any of these global symmetries. As such, they are accidental symmetries of \( L(g) \).

The conservation of \( B \) and \( L \), or their possible breaking only by non-
renormalizable interactions weighted by a high mass scale, is a welcome result. The
same cannot be said, however, for \( B_A \) and \( L_A \), since they are for example inconsistent with any mass term for quarks and leptons. Something has to be done to \( L(g) \) to cure this problem.

Before addressing this question, let us complete the description of the
fermionic degrees of freedom by introducing the two other replicas of the first
generation of fermions. Since these replicas have the same gauge interactions as the
first one, to the best of our present knowledge, this is simply done formally by
introducing an index \( i = 1,2,3 \) to \( \Psi \), which then becomes a triplet of column vectors,
\( \Psi_i \), one per generation. In turn, the fermion Lagrangian becomes

\[
\begin{align*}
L_{\text{fermion}}^{(g)} &= i \sum_i \bar{\Psi}_i D_i \Psi_i = i \sum_{\alpha, i} \bar{\alpha}_i D_\alpha \alpha_i.
\end{align*}
\]

As a net result, the accidental symmetry has become much larger than before, since
now we can also allow for independent unitary transformations on the index
\( i = 1,2,3 \), in the so-called "generation space", of any of the 5 irreducible
representations of the gauge group. Physically this corresponds to the fact that the
different generations are not distinct by the gauge interactions.

1.6 From \( L^{(g)} \) to a realistic Lagrangian

Before \( L^{(g)} \) can be called a realistic Lagrangian, two problems, at least, have to be solved: the breaking of the gauge symmetry and that of the accidental symmetries. Not to undermine the overall consistency of the theory, the gauge symmetry must be broken spontaneously [7]. On the contrary, the spontaneous breaking of the large global symmetry that we have discussed would be
phenomenologically problematic, because of the unwanted Goldstone bosons that would be introduced in this way.

The simplest solution of both these problems offered by the Standard Model is based on the introduction of a scalar particle, the Higgs doublet

\[ \varphi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix} \]  

of hypercharge \( Y = \frac{1}{2} \), transforming therefore under the gauge group as \( \varphi = (2, \frac{1}{2}) \), in the notation of section 1.3. As in the case of fermions, having assigned the transformation properties of \( \varphi \) under the gauge group fixes completely the form of its minimal gauge interactions

\[ L^{(g)} = |D_\mu \varphi|^2, \quad D_\mu = \partial_\mu - igW^i_\mu \sigma^i_j - ig^i B_\mu. \]  

There is, however, now an importance difference. The introduction of \( \varphi \) allows several more interaction terms to be written down, which are both gauge invariant and renormalizable. Taking precisely gauge invariance as the guiding principle, the Lagrangian of the SM is constructed by allowing all of them to be present. Putting together Eqs. (1.11) and (1.17) and adding the other allowed interactions, one gets the full Lagrangian [2]

\[ L = -\frac{1}{4} \text{Tr} W_{\mu \nu} W^{\mu \nu} - \frac{1}{4} B_{\mu \nu} B^{\mu \nu} + i \sum_i \bar{\Psi}_i D^i \Psi_i + |D_\mu \varphi|^2 - V(\varphi) + L_Y \]  

where \( V(\varphi) \) is a potential term in the \( \varphi \) field

\[ V(\varphi) = -\frac{\mu^2}{2} |\varphi|^2 + \frac{\lambda^4}{4} |\varphi|^4 \]  

and \( L_Y \) is a sum of Yukawa interactions of all the fermions with \( \varphi \)

\[ L_Y = Q_L \lambda^u_{ij} \varphi + Q_L \lambda^d_{ij} \varphi^* + L_L \lambda^e_{ij} \varphi^* \]  

Both in Eq. (1.19) and, even more so, in Eq. (1.20) we have left understood the contractions over the SU(2)xU(1) and Lorentz indices. The reader is invited to reconstruct explicitly these contractions, especially in order to appreciate that indeed
Eqs. (1.19) and (1.20) are all the extra terms that can be added to the Lagrangian. Of course we are neglecting all possible non-renormalizable interactions, but we have explained why in the previous section. On the other hand, we have made explicit in the Yukawa couplings the generation indices. The Yukawa couplings themselves depend on three arbitrary matrices in generation space, $\lambda^u, \lambda^d, \lambda^e$.

1.7 On global symmetries again

Is the introduction of $\phi$ effective in curing the problems that have been alluded to at the beginning of the previous section? Let us first look again at the global symmetries of the full Lagrangian in Eqs. (1.18-20).

At first sight, it would seem that the Yukawa interactions in Eq. (1.20), for arbitrary matrices $\lambda^u, \lambda^d, \lambda^e$, are there to break explicitly all of the accidental symmetries of $L^{(e)}$ except baryon and lepton number. This conclusion is, however, not quite correct. To see this explicitly, let us consider the diagonalization of the matrices $\lambda^d, \lambda^e$ by two independent bi-unitary transformations

$$\lambda^d = D^T \lambda^d_D D^c \quad \text{and} \quad \lambda^e = E^T \lambda^e_D E^c,$$

where $\lambda^d_D, \lambda^e_D$ are real diagonal matrices and $D, D^c, E, E^c$ are four independent unitary matrices. Remember that all these are matrices in flavour space. It is now possible to perform a set of unitary transformations defined on the matter multiplets as follows

$$D^c d^c \rightarrow d'^c, \quad DQ_L \rightarrow Q'_L, \quad E^c e^c \rightarrow e'^c, \quad EL_L \rightarrow L'_L.$$

These transformations are symmetries of $L^{(e)}$. As such, they leave it untouched. On the other hand, they send the Yukawa Lagrangian into

$$L_Y \rightarrow Q'_L \lambda'^u_{ij} u_j^c \phi + Q'_L \lambda^d_{di} d_i^c \phi^* + L'_L \lambda^e_{di} e_i^c \phi^* \quad \text{,} \quad \lambda' \equiv D^* \lambda^u.$$

The transformations in Eqs. (1.22) are canonical transformations which do not change the physics. They allow however a more transparent view of the global symmetries. Since leptons only occur in a diagonal sum over generations, not only the overall lepton number is conserved, but also the individual ones, $L_e, L_\mu, L_\tau$. On the contrary, the fact that we are left with an arbitrary matrix in the first term on the right hand side of Eq. (1.23) is the source of flavour non-conservation in the quark sector. The origin of the asymmetry between quarks and leptons is also clear: it can
be traced back to the fact that we did not have to introduce the right-handed neutrinos to describe the phenomenological weak interactions.

In conclusion we have discovered that the gauge theory based on the SU(2)xU(1) gauge group, with scalar and fermion matter fields transforming as specified above, has four accidentally conserved charges: overall baryon number and the individual lepton numbers. This is a very significant result, that agrees with observations. To the best of the present experimental knowledge, these are all absolute conservation laws. As to baryon number, the main implication is the stability of the proton, with a mode-dependent lower limit on its lifetime of about $10^{31}$ years [1]. On the individual lepton numbers, one can quote several limits. Probably the most significant among them are the following:

$$BR(\mu \rightarrow e\gamma) \leq 4.9 \cdot 10^{-11} \quad [8]$$

$$\frac{\sigma(\mu + Ti \rightarrow e + Ti)}{\sigma(\mu + Ti \rightarrow \nu + Ti)} \leq 10^{-12} \quad [9]$$

$$BR(\tau \rightarrow \mu \gamma) \leq 4.2 \cdot 10^{-6} \quad [10].$$

Finally, insisting on not introducing the right-handed neutrino, the upper bounds on the neutrino masses are also limits on the violation of lepton number, because any mass term, necessarily of the form $v_{Li} v_{Lj}$, carries two units of overall lepton number $L = L_e + L_\mu + L_\tau$ for any $i,j$. Notice that the reactions in Eqs. (1.23), if detected, would signal violations of individual lepton numbers, but not of $L$.

In conclusion, the discussion of the last two sections should have made clear that the current view of the electroweak interactions does not imply the absolute conservation of baryon and individual lepton numbers. The fact that they are absolutely conserved by the renormalizable Lagrangian of the SM simply furnishes an elegant rationale for the phenomenological suppression of any possible violation of them.

1.8 Breaking the gauge symmetry

Let us turn now to the breaking of the gauge symmetry. The key role is played in this case by the Higgs potential in Eq. (1.19). Taking $\mu^2$ positive gives to $V(\phi)$ the Mexican-hat form which induces a non vanishing expectation value (vev) on $\phi$. Being invariant under the gauge transformations, $V(\phi)$ depends only on $|\phi|^2$. The minimum of the potential occurs at
\[ \langle |\phi|^2 \rangle = \frac{\mu^2}{\lambda}, \quad (1.24) \]

with no preferred direction inside the doublet. The physics cannot actually depend on such direction. The only general consequence of Eq. (1.24) is that the gauge group is broken down to a U(1) subgroup of phase transformations, which can always be interpreted as the electromagnetic gauge invariance.

To see this, let us choose the vev of \( \phi \) as

\[ \langle \phi \rangle = \sqrt{\frac{\mu^2}{\lambda}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (1.25) \]

Notice that not only the direction but also the phase of \( \langle \phi \rangle \) can be fixed without any loss of generality, using the U(1) invariance of the original Lagrangian. We now ask: out of the gauge transformations, which are those that leave invariant the vev (1.25) and are therefore still invariances of the Lagrangian even after the replacement of \( \phi \) by \( \langle \phi \rangle \)? Only the transformations generated by \( T_3 + Y \) are, since this is the only combination of the four generators that annihilates \( \langle \phi \rangle \), being \( (T_3 + Y) \langle \phi \rangle = 0 \). \( Q = T_3 + Y \) is the residual unbroken gauge charge.

The main consequence of the breaking of gauge invariance is the appearance of a mass for the vector bosons corresponding to the broken generators and for all fermions except the neutrinos. The mass terms occur after we insert \( \langle \phi \rangle \) in the Lagrangian (1.18). For the vector bosons we have

\[ L^{(vb)}_m = |D_\mu \langle \phi \rangle|^2 = v^2 \left[ g W^i_\mu \left( \frac{\sigma^j}{2} + \frac{g'}{2} B_\mu \right) \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix}^2 = \]

\[ = v^2 \left[ \frac{g^2}{2} W^+_\mu W^-_\mu + \frac{1}{4} \left( W^3_\mu, B_\mu \right) \begin{pmatrix} g^2 & gg' \\ gg' & g'^2 \end{pmatrix} \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} \right], \quad (1.26) \]

where \( v = \sqrt{\frac{\mu^2}{\lambda}} \) and \( W^\pm_\mu = \frac{1}{\sqrt{2}} \left( W^1_\mu \pm iw^2_\mu \right) \).

This is a mass term both for the charged \( W \) boson, \( M_W^2 = \frac{g^2 v^2}{2} \), and for the neutral bosons. Actually, the 2X2 mass matrix of the neutral bosons has only one non-zero
eigenvalue, \( M_Z^2 = \frac{(g^2 + g'^2)v^2}{2} \), as readily seen by making the rotation in the space of the neutral bosons

\[
\begin{pmatrix}
A_\mu \\
Z_\mu
\end{pmatrix} = 
\begin{pmatrix}
\sin \vartheta & \cos \vartheta \\
\cos \vartheta & -\sin \vartheta
\end{pmatrix}
\begin{pmatrix}
W^3_\mu \\
B_\mu
\end{pmatrix}
\]  

(1.27)

where

\[\tan \vartheta = \frac{g'}{g}.\]  

(1.28)

Being mass eigenstates, \( A_\mu \) and \( Z_\mu \), unlike \( W^3_\mu \) and \( B_\mu \), are the physical neutral bosons, of mass 0 and \( M_Z \), respectively. The presence of a massless eigenvalue is a direct consequence of the residual \( U(1) \) gauge invariance.

The physical meaning of the various vector boson mass eigenstates is given by their interactions with the fermions, which acquire the form, using Eqs. (1.27, 28),

\[
L_I = -i\overline{\psi}_\mu \left( gW^i_\mu T^i + g'YB_\mu \right) \psi = 
- i\overline{\psi}_\mu \left( gW^+_\mu T^+ + g W^-_\mu T^- + g \sin \vartheta QA_\mu + g \cos \vartheta \left( T_3 - \sin^2 \vartheta Q \right) Z_\mu \right) \psi .
\]  

(1.29)

\( L_I \) contains, other than the charged current interactions, the electromagnetic interaction, requiring the identification of \( g \sin \vartheta \) with the electron charge \( e \), and the neutral current interaction mediated by the Z-boson.

The masses of the fermions come from the Yukawa interaction Lagrangian in Eq. (1.23)

\[
L^f_m = Q_{Li}\lambda^u_{ij}u^c_j \phi + Q_{Li}\lambda^d_{Di}d^c_i \phi + \lambda^e_{Di}c^c_i \phi = 
= u_{Li}\lambda^u_{ij}u^c_j + d_{Li}\lambda^d_{Di}d^c_i + e_{Li}\lambda^e_{Di}c^c_i
\]  

(1.30)

where we have suppressed the irrelevant primed indices. Notice that, with an appropriate definitions of the fermion fields, Eqs.(1.22), we have made diagonal both the mass terms for the down-type quarks and for the charged leptons but not yet for the up-type quarks. We shall come back to this problem in the next Chapter.

1.9 On the \( \rho \) parameter.
We have noticed in the previous section that the mass matrix of the neutral vector bosons has a vanishing eigenvalue due to electromagnetic gauge invariance. Furthermore, the non zero eigenvalue is related to the mass of the charged boson by

\[ M_W^2 = \cos^2 \theta M_Z^2 \]  

(1.31)

where \( \theta \) is the same angle, defined in Eq. (1.28), which enters into the interaction Lagrangian (1.29). It is customary to define a dimensionless parameter, \( \rho \), such that, at the tree level,

\[ \rho = \frac{M_W^2}{M_Z^2 \cos^2 \theta}. \]  

(1.32)

Up to radiative correction effects, the SM predicts therefore that \( \rho = 1 \) [11]. The interesting aspect of this relation, or equivalently of Eq. (1.31), is that it is not, unlike the vanishing of one eigenvalue of the neutral boson mass matrix, a general consequence of gauge invariance and its partial breaking. To see this, the reader is invited to consider the spontaneous breaking of SU(2)xU(1) induced, e.g., by a triplet Higgs field, rather than by a doublet as in Eq. (1.25). On the other hand, \( \rho = 1 \) is phenomenologically a very successful relation, as we shall see when we discuss the radiative correction effects. As such, it deserves a special comment.

The origin of \( \rho = 1 \) in the SM can be traced back to a symmetry of the Higgs potential (1.19) which is larger than the gauge symmetry itself [12]. To see this explicitly, let us construct, out of the \( \phi \) field, a 2X2 matrix

\[ \Phi = \begin{pmatrix} \phi^0 & \phi^- \\ -\phi^+ & \phi^0 \end{pmatrix} \]  

(1.33)

whose columns are the vectors \( i\sigma_2 \phi^* \) and \( \phi \) itself. Since \( Tr(\Phi^\dagger \Phi) = 2|\phi|^2 \), the potential (1.19) is a function of \( Tr(\Phi^\dagger \Phi) \) and it has the full invariances of this trace, which are

\[ \Phi \rightarrow U_L \Phi U_R^\dagger \]  

(1.34)

with \( U_L \) and \( U_R \) arbitrary and independent unitary 2X2 matrices. This means that, apart from the hypercharge phase invariance, the potential is actually symmetric under two independent SU(2) transformations. Only the one acting on the left is the
gauged SU(2) invariance, since also \( i\sigma_2 \varphi^* \) transforms as a doublet if \( \varphi \) does. This larger invariance of the potential has everything to do with the fact that \( |\varphi|^2 = \sum_i \varphi_i^2 \), the sum being extended over the four real components of \( \varphi \), so that the modulus of \( \varphi \) is actually invariant under a generic rotation of all the four components among each other. These rotations make the group SO(4), which is locally isomorphic to SU(2)XSU(2). The important point about this symmetry group is that, after symmetry breaking, namely after \( \varphi \) has got a non vanishing vev, SU(2)XSU(2) gets broken down to the so called diagonal SU(2), with \( U_L \) and \( U_R \) being this time identical. This follows from the fact that the vev of the matrix \( \Phi \), is proportional to unity, \( \langle \Phi \rangle = v I \).

What does all of this have to do with the masses of the vector bosons? Consider the covariant kinetic term of the Higgs field, which is the source of the boson masses, with the \( g' \) coupling constant switched off

\[
|D_\mu \varphi|^2 \rightarrow \left( \partial_\mu - ig W^i_\mu \frac{\sigma^i}{2} \right) \varphi^2.
\]

The point is that this term too has the full SU(2)XSU(2) invariance, with the \( W^i \) transforming as a triplet under \( SU(2)_L \) and as singlets under \( SU(2)_R \). This means that, after symmetry breaking, the W mass term must be invariant under the residual diagonal SU(2), which again acts on the \( W^i \) as a triplet, or

\[
L_m^{(W)} \propto \sum_i W^i_\mu^2 = 2W^+_\mu W^-_\mu + W^3_\mu W^3_\mu.
\]

This explains the relation in Eq. (1.26) between the charged W-mass and the 11 entry of the neutral boson mass matrix, which is responsible for \( p = 1 \).

We conclude this section with two considerations.

The phenomenological success of \( p = 1 \) might be a consequence of an enlarged symmetry property of the gauge symmetry breaking sector of the electroweak interaction Lagrangian, more general than the form that it takes with the explicit Higgs doublet realization characteristic of the SM [12]. This larger symmetry is often called "custodial" in the literature.

The fact that the SU(2)XSU(2) symmetry is actually not a symmetry of the full SM Lagrangian calls for deviations of \( p \) from 1 when the radiative corrections are included. This will be discussed in Chapter 3. Sources of breaking of the extra SU(2) symmetry are the \( g' \) coupling of the hypercharge U(1) as well as the fermion Yukawa couplings, in so far as they distinguish the up from the down component of
the SU(2) doublets. In fact, e.g., the top-bottom Yukawa couplings, the largest and therefore the most important ones, can be written as

\[ L_Y^{(t-b)} = Q_L^c \Phi^* \left( \begin{array}{c} \lambda^t \\
\lambda_b 
\end{array} \right) Q_c + h.c. \tag{1.37} \]

where the singlets under \( SU(2)_L \) are organised in a doublet under \( SU(2)_W \) so that

\[ Q^c \equiv \left( \begin{array}{c} f^c \\
b^c 
\end{array} \right) \xrightarrow{SU(2)_W} U_R Q^c. \tag{1.38} \]

The transformation (1.38) would indeed be a symmetry of \( L_Y^{(t-b)} \) if it were \( \lambda^t = \lambda_b \), which is of course badly violated.

1.10 Neutral currents

As shown in section 1.2, with regard to the interaction of fermions with the vector bosons (VB), the main phenomenological implication of extending the Fermi theory to a full gauge theory is the existence of neutral currents, mediated by the exchange of the Z boson. The observation of the neutral currents by Gargamelle in 1973 was indeed the first experimental verification of the SM [13]. Subsequently, parity violation phenomena in atoms and in deep inelastic scattering off nuclei of polarised electrons played a crucial role. More recently, experiments in \( e^+ - e^- \) at the Z-pole can be viewed as a triumph of the theory of neutral currents.

Muon neutrino scattering off electrons deserves some special comments: in many respects it is the simplest and best defined manifestation of neutral currents and has also been the first process to be identified in Gargamelle.

In terms of the Z-coupling to the electron, as obtained from Eq. (1.29),

\[ V_\mu(Z \to e\bar{e}) = -g \cos \theta \bar{\psi}_\mu (g_{Ve} - g_{\Delta e} Y_5) e, \]

\[ g_{Ve} = -\frac{1}{2} + 2 \sin^2 \theta, \quad g_{\Delta e} = -\frac{1}{2}, \tag{1.39} \]

the total cross section for \( v_\mu - e \) scattering is given by

\[ \sigma_{v_\mu} = \sigma_0 \left[ 4 g_L^2 + \frac{4}{3} g_R^2 \right], \quad \sigma_0 = \frac{G^2 m_e E_V}{2\pi} = 4.3 \cdot 10^{-42} \text{ cm}^2 \left( \frac{E_V}{\text{GeV}} \right), \tag{1.40} \]

\[ g_L = \frac{g_{Ve} - g_{\Delta e}}{2}, \quad g_R = \frac{g_{Ve} + g_{\Delta e}}{2}. \tag{1.41} \]
The cross section for $\bar{\nu}_\mu - e$ scattering is obtained from the substitutions:

$$\sigma_{\nu_e} \rightarrow \sigma_{\bar{\nu}_e} : \ g^2_{L,R} \rightarrow g^2_{R,L}.$$  \hspace{1cm} (1.42)

On the other hand, the electron neutrino or antineutrino cross-sections, always off electrons, receive an additional contribution from a charged $W$-boson exchange in the crossed channel, a crucial difference for the theory of neutrino oscillations in matter. This fact is accounted for in the cross sections by replacing $g_L$ with $g_L + 1$.

The combination of the four experimental cross sections for $\nu_\mu - e$, $\bar{\nu}_\mu - e$, $\nu_e - e$ and $\bar{\nu}_e - e$ define two possible determinations of $g_L$ and $g_R$, due to the symmetry of all cross sections under the substitution: $g_L \rightarrow g_R$ and $g_R \rightarrow -g_L$. A unique solution is obtained by requiring $g^A_e = \frac{1}{2}$, as implied by the forward backward asymmetry in $e^+ - e^- \rightarrow \mu^+ - \mu^-$ at low energy. There has in fact been a remarkable progress in the last twenty years in the measurements of the neutrino electron cross sections. What these cross sections actually measure are the combinations $g^v_{A,e} = 2g^v g_{\nu_e,Ae}$, where $g^v$ is the coupling of the (left handed) neutrino to the $Z$. $Z$-decays, on the other hand, give directly $g_{\nu_e,Ae}$. Comparing the two sets of results, we may test the SM prediction $g^v = \frac{1}{2}$.

Combining data on $Z$-decays and neutrino cross-section data from CHARM II [14], one obtains $|2g^{\nu_{\mu}}| = 1.006 \pm 0.036$. LEP results alone measure the same quantity from the determination of the $Z$-width into invisible channels (neutrinos in the SM). Assuming lepton universality, one obtains $|2g^{\nu_{\mu}}| = 1.006 \pm 0.006$ [15], which gives a clear idea of the relative sensitivities.

1.11 Lepton universality (in charged currents)

In the previous section we have discussed the main phenomenological consequence of extending the Fermi theory of the weak interactions to a full gauge theory as far as the fermion interactions are concerned: the existence of neutral currents is the main prediction of the tree level Lagrangian of the SM. On the other hand, the great progress represented by the SM is of course that it allows a consistent treatment of the electroweak interaction phenomena to all orders in perturbation theory[16]. To achieve this purpose has in fact been a basic guiding line at all in the construction of the SM. The phenomenological impact of this aspect of the theory will be discussed at length in Section 3, where we introduce the subject of the
radiative corrections and we illustrate the precision tests of the SM. It is important to realise, however, that there are some observables in electroweak physics which can be predicted with a high level of accuracy without resorting to the full machinery of the electroweak radiative corrections. Lepton universality in charged currents is an example of this type of predictions. We have in mind the comparison of the \( \tau \) and \( \mu \) leptonic decays as well as of the charge pion decays into electrons and muons.

Let us start from \( \tau \) and \( \mu \) leptonic decays. Calling \( L = \tau, \mu \) and \( l = \mu, e \) the charged leptons in the initial and final state of mass \( m_L \) and \( m_l \) respectively, one has, at the tree level from W exchange,

\[
\Gamma^{(0)}(L \to l\nu_L\bar{\nu}_l) = \frac{G^2 m_L^5}{192 \pi^2} \left( 1 + \frac{3}{5} \frac{m_L^2}{M_W^2} \right) f \left( \frac{m_L^2}{m_\tau^2} \right)
\]

(1.43)

where the last factor in the right hand side accounts for the phase space dependence on the final lepton mass, normalised to 1 at \( m_l = 0 \). How about the radiative corrections to this formula, due to virtual photon, W and Z exchanges? The important point about them is that one can talk of the pure electromagnetic corrections (photon exchanges) in isolation from the weak ones (W and Z exchanges). This comes about as follows.

In the Fermi approximation, with the W propagator contracted to a point, the effective Hamiltonian responsible for the decay can be written as

\[
H_{\text{eff}}^0 = \frac{G}{\sqrt{2}} \left( \bar{\nu}_L \gamma_\mu (1 + \gamma_5) \nu_L \right) \frac{1}{L} \left( \bar{\nu}_L \gamma_\mu (1 + \gamma_5) \nu_L \right)
\]

(1.44)

To get the last form of \( H_{\text{eff}}^0 \), we have performed a Fierz rearrangement between the initial \( L \) and the final \( \bar{\nu}_l \). The advantage of this, so called, "charge retention" form, is that it makes clear that the photon-exchange corrections can be viewed as a renormalization of the vector and axial \( L \to l \) transition currents \( \bar{\nu}_L \gamma_\mu (1 + \gamma_5) L \). By a well known theorem [17], that the reader is invited to check explicitly on the one loop diagrams, vector and axial currents suffer at most finite, i.e. cut-off independent, corrections. There is no divergent piece in the pure electromagnetic corrections, which introduce a correction factor to Eq. (1.43), [18]
\[ \Gamma(L \to l\nu_L\bar{\nu}_l(\gamma)) = \Gamma^{(0)}(L \to l\nu_L\bar{\nu}_l) \left(1 + \frac{\alpha(m_L)}{2\pi} \left(\frac{25}{4} - \pi^2\right)\right). \] (1.45)

readily computable without any knowledge of the full SM Lagrangian. Of course there are also pure weak corrections to Eq. (1.45), due to W and Z exchanges, but, being also finite, they are of universal nature and can be lumped in the Fermi constant \( G \), (see Chapter 3), apart from negligible terms vanishing as \( m_\mu^2/M_\mu \).

The upshot of all this is that Eq. (1.45) is a very accurate formula, which can be compared with experiments. Using the muon lifetime to determine \( G \) [1], one predicts

\[
\begin{align*}
\text{BR}(\tau \to e\nu_\tau\bar{\nu}_e(\gamma)) &= 0.18116 \left(\frac{m_\tau}{1777.1 \text{MeV}}\right)^5 \left(\frac{\tau_\tau}{295.6 \text{fs}}\right), \\
\text{BR}(\tau \to \mu\nu_\tau\bar{\nu}_\mu(\gamma)) &= 0.17620 \left(\frac{m_\tau}{1777.1 \text{MeV}}\right)^5 \left(\frac{\tau_\tau}{295.6 \text{fs}}\right)
\end{align*}
\] (1.46a, 1.46b)

with the \( \tau \) mass and lifetime normalised to their 1994 PDG values [1]. The present experimental determinations of the same quantities are [1]

\[
\text{BR}(\tau \to e\nu_\tau\bar{\nu}_e(\gamma))_{\text{exp}} = (18.01 \pm 0.18)\% \quad \text{and} \quad \text{BR}(\tau \to \mu\nu_\tau\bar{\nu}_\mu(\gamma))_{\text{exp}} = (17.88 \pm 0.24)\%. 
\]

A partially similar situation is encountered in discussing the ratio of the electronic to muonic decays of the pion, which is given by [19]

\[
R = \frac{\text{BR}(\pi \to e\nu_e(\gamma))}{\text{BR}(\pi \to \mu\nu_\mu(\gamma))} = \left(\frac{m_e}{m_\mu}\right)^2 \left(1 - \frac{m_e^2}{m_\pi^2}\right)^2 \left(1 + \frac{3\alpha}{\pi} \ln \frac{m_e}{m_\mu} + O(\alpha)\right) = 1.235 \cdot 10^{-4}. \] (1.47)

In this case, it is in taking the ratio that one gets rid of a logarithmic divergence in the electromagnetic corrections, (which remain finite and calculable, including the \( O(\alpha) \) term, not written down explicitly for brevity) as well as of uncertainties due to the
strong interactions. Also in this case, Eq. (1.47) compares favourably with the present experimental results:

\[
R_{\text{exp}} = \begin{cases} 
(1.2265 \pm 0.0034 \pm 0.0044) \cdot 10^{-4} [\text{TRIUMF}] \\
(1.2346 \pm 0.0035 \pm 0.0036) \cdot 10^{-4} [\text{PSI}] 
\end{cases}
\]
2. FLAVOUR PHYSICS

2.1 Flavour changing interactions. $V_{\text{CKM}}$

We have already seen in section 1.5 that the electroweak interactions do not conserve the individual quark-flavour numbers. We have also seen that in the original SM Lagrangian, working in the basis defined by Eqs. (1.22), flavour violation is confined to the arbitrary matrix of the Yukawa couplings for the up-type quarks, $\lambda^u$. For a more transparent physical definition of the flavour changing interactions, one has to work, however, in the mass eigenstate basis for the quarks, which allows the very concept of flavour distinction to make sense at all. Remember that the fermion mass terms in Eq. (1.30) were already diagonal for the $Q = \frac{2}{3}$ quarks and for the leptons, but not yet for the quarks of charge $Q = \frac{1}{3}$. This is easily remedied by an appropriate redefinition of the up-type quark fields. Defining, in analogy with Eqs. (1.21), $\lambda^u \equiv U^T \lambda^u_D U^c$, this diagonalization is achieved by the field redefinitions

$$Uu_L \rightarrow u'_L, \quad U^c u^c \rightarrow u'^c. \quad (2.1)$$

The question is, at this point: Where is the flavour violation gone, now that even the up-quark mass matrix has been diagonalized and, thereby, all quark fields are physical mass-eigenstates? It must be present in the interactions, which we now discuss.

All the fermion interactions in the SM are those with the VB and with the physical Higgs field $H$, proportional to the mass terms. Before performing the unitary transformations (2.1), they were respectively given by

$$L^8_j = -i \bar{\psi} \gamma_{\mu} \left( g W^+_{\mu} T^- + g W^-_{\mu} T^+ + g \sin \theta Q A_{\mu} + g \cos \theta \left(T_3 - \sin^2 \theta Q\right) Z_{\mu}\right) \psi$$

and

$$L^H_j = H \left( u_{Lj} \lambda^u_{ij} u^c_j + d_{Lj} \lambda^d_{ij} d^c_j + e_{Lj} \lambda^e_{ij} e^c_j \right). \quad (2.2)$$

If we now go to the physical up-type quark basis, via (2.1), the Higgs interactions become flavour diagonal. Likewise, the photon- and the $Z$-couplings, being proportional to neutral diagonal generators, $Q$ and/or $T_3$, are not affected by the unitary transformations (2.1) and remain perfectly flavour diagonal, as the fermion
kinetic terms do. On the contrary, the charged current interacting with the $W$, which connects quarks of different charge, becomes non-diagonal in flavour:

$$L_C^e = -iW^\mu \left( \bar{u}_{Li} U^{+\mu ij} \gamma_\mu d_{Lj} \right) + h.c. \quad (2.4)$$

where $U$ is one of the unitary matrices in (2.1) and we have again suppressed the irrelevant primed indices. This is a main physical result, characteristic of the SM: In the quark physical basis, it is only in the charged current interaction (2.4) that all the flavour changing interactions reside [20]. The unitary 3X3 matrix $U^+$ in (2.4) is the Cabibbo-Kobaishi-Maskawa (CKM) matrix, usually denoted by $V_{CKM}$.

The CKM matrix plays a crucial role in the physics of flavour. A self-explanatory notation for it is

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \quad (2.5)$$

More important is to fix a parametrization for $V_{CKM}$ in terms of a minimum number of physical real parameters. Let us discuss such parametrization, having in mind the possibility of $n$ generations, rather than 3. A $n \times n$ unitary matrix (an element of the group $U(n)$) has $n^2$ arbitrary real parameters, whereas a $n \times n$ orthogonal matrix (an element of $SO(n)$) has $\frac{1}{2} n(n-1)$ real parameters. Since a real unitary matrix is an orthogonal matrix, it follows that a $n \times n$ unitary matrix has $\frac{1}{2} n(n+1) = \left[ n^2 - \frac{1}{2} n(n-1) \right]$ phases. Not all of these phases, however, have physical meaning, since some of them can be removed by a canonical field redefinition. To count the uneliminable, or physical, phases is essential, since any such phase in a parameter entering the Lagrangian of a local quantum field theory is a necessary and sufficient condition for CP violation. In the SM, it is in the physical phases of the CKM matrix that the source of CP violation can reside.

To this purpose, we have to remember that the CKM matrix enters into the charged current, $J^\mu = \bar{u}_{Li} (V_{CKM})_{ij} \gamma_\mu d_{Lj}$, which involves $2n$ independent fields. Their phase redefinitions allows to multiply the elements $(V_{CKM})_{ij}$ by $(2n - 1)$ independent phases, which do not show up anywhere else in the Lagrangian. They would actually do in the fermion masses or in the Higgs interactions (2.3), but they can be compensated by a redefinition of the fields $u^e_i, d^e_i$, which do not appear in the...
charged current interactions. We are therefore left with 
\[ \frac{1}{2}(n-1)(n-2) = \frac{1}{2}n(n+1) - (2n-1) \] physical phases. With 3 generations, this gives one physical phase, which is just enough to describe CP violation [21].

The parametrization of the CKM matrix chosen by the PDG in terms of four real physical parameters is

\[
V_{CKM} = \begin{pmatrix}
    c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
    -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
    s_{12}c_{13} - c_{12}s_{23}s_{13}e^{i\delta} & -c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix}.
\] (2.6)

### 2.2 Quark-lepton universality. \( V_{ud}, V_{us} \)

The universality of the couplings of quarks and leptons is a classic subject in weak interactions, which has played a key role in the construction of the theory since the beginning. In its modern version, the problem is the one of controlling how the unique gauge coupling \( g \) appearing in front of the charged weak interaction Lagrangian, reflects itself in a correlation of the strength of processes involving quarks with processes involving leptons. The description of this subject goes through several logical steps which are common to the discussion of many problems in weak interaction physics and that will be often encountered in the rest of this Chapter.

i) The Effective Weak Hamiltonian. The masses and momenta relevant to the processes of interest in this problem, (generically called \( \mu \)), are low compared to the characteristic scale of the weak interactions. Looking for the leading term in an expansion in inverse powers of this scale, we consider the lowest order relevant diagrams mediated by \( W \) or \( Z \) exchanges and we contract their propagators to a point, a' la Fermi. In this way we obtain the *bare effective Hamiltonian* for weak interactions as

\[
H_{\text{eff}}^{(0)} = \sum_i c_i O_i
\] (2.7)

where \( O_i \) are 4-fermion local operators. In Chapter 1 we have considered examples of \( H_{\text{eff}}^{(0)} \), when discussing the \( \beta \)-decay of the neutron

\[
H_{\text{eff}}^{(0)} = \frac{G}{\sqrt{2}} V_{ud} \left( \bar{u} \gamma_\mu (1 + \gamma_5)d \right) \left( \bar{e} \gamma_\mu (1 + \gamma_5)v_e \right)
\] (2.8)
(with \( \cos \theta_C \) replaced by the now more appropriate \( V_{ud} \)) or lepton universality

\[
H_{\text{eff}}^{(0)} = \frac{G}{\sqrt{2}} \left[ \bar{f} \gamma_\mu (1 + \gamma_5) f \right] \left[ \bar{\nu}_L \gamma_\mu (1 + \gamma_5) \nu_L \right].
\] (2.9)

The various \( Q_i \) are in particular characterised by different selection rules in flavour.

ii) Renormalization Group rescaling of the Effective Hamiltonian [22]. Radiative corrections to the tree level \( W \) and \( Z \) exchanges that give rise to the bare effective Hamiltonian have to be considered. Of special importance are those involving the photons and the gluons, since they can be infrared divergent when the external masses and momenta are neglected. In such a case, in fact, the expansion parameter is \((\alpha, \alpha_5) \ln \left( \frac{\Lambda^2}{\mu^2} \right)\) rather than \( \alpha \) or \( \alpha_5 \).

This problem is dealt with by considering the four-fermion point-interactions corresponding to the \( Q_i \) and by dressing them with all possible one gluon or one photon exchanges. The diagrams so obtained have to be computed with an ultraviolet cut-off \( \Lambda \). If we choose for the \( Q_i \) a basis of operators which are multiplicatively renormalized, the result of such calculation has the form (we discuss only electromagnetic corrections for concreteness)

\[
Q_i^{\text{ren}} = \left( 1 + \alpha_i \frac{\gamma_i}{\ln \Lambda^2} \right) Q_i^{\text{bare}}.
\] (2.10)

We have in fact shown in section 1.11 that no such divergences are present in the case of the operator in Eq. (2.9). For completely analogous reasons there are no divergent strong corrections to the operator in Eq. (2.8), (no divergent renormalization of the vector or axial quark currents). There exist, on the contrary, divergent electromagnetic corrections to the \( \beta \)-decay operator. The scale of the ultraviolet cut-off corresponds to the \( W \) mass, which has been taken to infinity in the bare Hamiltonian.

The Renormalization Group (RG) now states that the effective weak Hamiltonian at a scale \( \mu << M_W \) is expressed in the following way

\[
H_{\text{eff}} = \sum_i c_i(\mu) Q_i(\mu),
\] (2.11)

where the coefficients \( c_i(\mu) \) satisfy the RG Equations.
\[ \frac{-\partial}{\partial t} + \beta(\alpha) \frac{\partial}{\partial \alpha} + \Gamma_i(\alpha) \] \quad c_i(\mu) = 0, \quad t = \ln \left( \frac{M_W^2}{\mu^2} \right), \quad (2.12) \]

and \( O_i(\mu) \) are the four-fermion operators renormalized at the scale \( \mu \) to free field operators. In the RGE, \( \beta(\alpha) \) and \( \Gamma_i(\alpha) \) are the coupling constant, \( \beta \)-function, \( \beta(\alpha) = \beta_0 \alpha^2 + \ldots \), and the anomalous dimension of the operators \( O_i, \Gamma_i(\alpha) = \alpha \cdot \gamma_i + \ldots \), with the \( \gamma_i \) defined in (2.10). The solution of Eq. (2.12) for the coefficients \( c_i(\mu) \) has the form

\[ c_i(\mu) = c_i(M_W, \alpha(M_W)) \exp \int_0^t \Gamma(\alpha(t)) dt' = c_i \left[ \frac{\alpha(M_W)}{\alpha(\mu)} \right]^{Y_i}. \quad (2.13) \]

In the case of interest for the \( \beta \)-decay operator, the effective Hamiltonian in (2.8) gets therefore renormalized to

\[ H_{\text{eff}} = \frac{G}{\sqrt{2}} V_{ub} \left[ \frac{\alpha(M_W)}{\alpha(\mu)} \right]^{Y_i} b_0 (\bar{u} i \gamma_\mu (1 + \gamma_5) d) (\bar{e} i \gamma_\mu (1 + \gamma_5) v_e). \quad (2.14) \]

with \( \gamma = \frac{1}{2\pi}, b_Q = \sum \frac{Q_f^2}{3\pi} \) (the sum being extended over the charges of all fermions with a mass between \( \mu \) and \( M_W \)). In this way, Eq. (2.14) resums all powers of \( \alpha \ln M_W \) in the electromagnetic corrections to the bare Hamiltonian. With respect to the bare Hamiltonian (2.8), we have found a correction factor, which, taking

\[ \alpha(\mu = 1 GeV) = \frac{1}{135} \quad \text{and} \quad \alpha(M_W) = \frac{1}{129}, \quad (2.15) \]

is numerically 1.026. A more precise treatment, involving the rescaling in several steps corresponding to the thresholds of the various charged-particle masses, gives a slightly lower value for the correction factor. (For a discussion on the running of the electromagnetic coupling, see Chapter 3). To achieve this, it is important to realise that we did not have to make any reference to the full Lagrangian of the SM, since we have simply resummed the leading-logarithmic electromagnetic corrections. We would have needed the SM Lagrangian to make precise the identification of the ultraviolet cut-off, which was taken at \( M_W \), but could have been any finite factor times \( M_W \) itself.
iii) Hadronic matrix elements. The final step necessary to get a complete prediction for the $\beta$-decay processes, requires taking the matrix element of the effective Hamiltonian (2.14), valid at low energies, between the desired initial and final states. In the case of the $\beta$-decay, this is trivial for the leptonic piece, but not so much for the hadronic part. To be precise, for the decay of a nucleus $N$ (e.g. the neutron) to a nucleus $N'$ (e.g. the proton), what we need is the matrix element of the quark charged weak-current

$$\langle N|\bar{u}\gamma_\mu(1+\gamma_5)d|N'\rangle.$$ \hspace{1cm} (2.16)

This is a special hadronic matrix element indeed, that makes it much more controllable than the typical matrix elements encountered in weak interaction problems and which are plagued by the well known and obvious strong interaction uncertainties. What makes (2.16) special, is the similarity of the matrix element of the vector piece of the weak current, the one with the $\gamma_\mu$ term, with the matrix element of the electromagnetic current: both currents are conserved by the strong interactions, exactly or to a good approximation. This is because, in Quantum Chromo Dynamics, $\bar{u}\gamma_\mu d$ is proportional to the current related to the $I^-\!$ component of isospin. Therefore, much in the same way as the normalisation of the matrix element of the electromagnetic current between hadronic states is controlled, at low momentum transfer, by the charge of the hadron

$$\langle N(p)|\frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d|N(p')\rangle \equiv (p+p')_\mu \langle N|Q|N\rangle,$$ \hspace{1cm} (2.17)

we have for the vector weak current

$$\langle N(p)|\bar{u}\gamma_\mu d|N'(p')\rangle \equiv (p+p')_\mu \langle N|I^-|N'\rangle.$$ \hspace{1cm} (2.18)

In those cases where a selection rules forbids any contribution to (2.16) from the axial current, e.g. in Fermi super-allowed $J^P = 0^+ \rightarrow 0^+$ nuclear transitions, Eq. (2.18) allows a precise control of the matrix element of interest.

An analogous discussion can be made for the matrix elements of the current $\bar{u}\gamma_\mu s$ involving a change of strangeness and relevant, for example, in hyperon $\beta$-decays or in $K_{I3}$ transitions. In this case, however, one has to rely on the approximate conservation, by strong interactions, of the SU(3) symmetry rather than the SU(2) of isospin. Or rather, more precisely, one has to correct the naive result.
analogous to Eq. (2.18) by taking into account the small but non negligible SU(3)-breaking effects.

iv) Collection of physical results. We are finally in the position to make contact with experiment. The comparison of a hadron semi-leptonic weak decay, controlled by the Hamiltonian (2.14) or by its \(|\Delta S|=1\) (or \(|\Delta B|=1\)) counterpart, with \(\mu\)-decay, controlled by (2.9), allows a determination of the relevant elements of the CKM matrix. This is a necessary step in order to check quark-lepton universality in charged weak currents. In the SM, quark-lepton universality in charged weak current is in fact nothing but the statement that \(V_{CKM}\) is unitary.

From "super-allowed" \(J^P = 0^+ \rightarrow 0^+\) nuclear transitions in a series of nuclei, from \(^{14}\text{O}\) to \(^{54}\text{Co}\), the full theoretical machinery described so far, actually supplemented by several more refinements, gives \([1]\)

\[
|V_{ud}|_{\text{nuc.trans.}} = 0.9744 \pm 0.0010
\]  

The error in Eq. (2.19) is theoretically dominated and is, however, too small to be fully believable.

The neutron life-time too is a source of information for \(V_{ud}\). In this case, also the matrix element of the axial weak current contributes to the decay, which cannot be normalised theoretically as effectively as the vector current. From the angular distribution of the electron in the decay of a polarised neutron, however, the relative strength of the axial to vector nucleon form factors, \(gA/gV\), can be directly measured. (See Eq. (1.3)). A combination of theory \([23]\) and experiment \([1]\) gives

\[
|V_{ud}|_{\text{neut.dec.}} = 0.9812 \pm 0.0022,
\]  

with an error this time dominated by the experimental uncertainty on \(gA/gV\). Notice in any case that the errors quoted in Eqs. (2.19,20) are clearly smaller than the size of the electromagnetic radiative-correction effect discussed above.

As mentioned, the sources of information for \(V_{us}\) are the \(\beta\)-decays of the hyperons and the \(K_{l3}\) decays, \(K^+ \rightarrow \pi^0 e^+ \nu\) and \(K_L \rightarrow \pi^+ e^- \nu\). The comparison of theory with experiment gives in this case \([1]\)

\[
|V_{us}| = 0.2205 \pm 0.0018
\]  

(2.21)
We are finally in the position to check the unitarity of the CKM matrix at the level of the first row, which is what one calls nowadays the test of quark-lepton universality. Note that the third element of interest, $V_{ub}$, is too small to be relevant. (See next section). We have, from Eqs. (2.19-21)

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = \left\{ \begin{array}{l}
(\text{from } 0^+ \rightarrow 0^+ \text{ nucl. trans.}) \\
0.9981 \pm 0.0020(\Delta V_{ud}) \pm 0.0008(\Delta V_{us}) \\
(\text{from neutron decay}) \\
1.011 \pm 0.0042(\Delta V_{ud}) \pm 0.0008(\Delta V_{us})
\end{array} \right..$$

(2.22)

As mentioned, the estimate of the theoretical error on $V_{ud}$ as obtained from nuclear $\beta$-transitions is likely to be too optimistic. In any case, Eq. (2.22) constitutes a significant quantitative test of the theory of weak interactions.

2.3 $b$-quark decays. $V_{cb}, V_{ub}$

The decays of the heavy $b$ quark are intensively studied as they provide the source of information for $V_{cb}$ and $V_{ub}$. At the time of writing these notes, the assessment of the theoretical uncertainties present in these studies is still controversial. We will therefore only outline the general subject.

The inclusive semi-leptonic width of a hadron $H_b$ containing a $b$-quark into any charmed final state $X_c$ can be written as

$$\Gamma (H_b \rightarrow X_c l\bar{\nu}_l) = \frac{G^2 m_b^5}{192 \pi^3} |V_{cb}|^2 f \left( \frac{m_c^2}{m_b^2} \right) \left[ 1 + O(\alpha_S) + O\left( \frac{\Lambda^2}{m_b^2} \right) \right].$$

(2.23)

where $m_b$ and $m_c$ are the bottom and charm quark masses and $\Lambda$ is the QCD scale. One should notice the similarity of this expression with the one, given in Eq. (1.43), for the leptonic decay width of a heavy lepton. Indeed, if one neglects the factor in squared parentheses, Eq. (2.23) gives the decay width in the free quark model; in particular this result has no reference at all to the initial hadron, which can for example be a meson or a baryon.

The rationale for this so-called "spectator quark model", a part from physical intuition, goes through the observation that the inclusive semi-leptonic decay of a heavy quark (of mass $m >> \Lambda$) can be treated in a fashion similar to deep-inelastic scattering (with a momentum transfer $q >> \Lambda$) [24]. In essence, up to terms vanishing with $\frac{\Lambda}{m_b}$, the effects of gluons connecting the charmed quark in the final state with the light spectator quark(s) in $H_b$ are negligible. Furthermore, as in deep-inelastic
scattering, the effects vanishing as powers of $\frac{\Lambda}{m_b}$, can be cast into matrix elements, between the initial state, of local operators, which can be obtained from other phenomenological sources [25]. As a consequence, and more precisely, the terms indicated as $O(\alpha_S^i)$ in the right-hand-side of Eq. (2.23) can be computed in the parton model, much in the same way as one gets the QED correction factor in the heavy lepton decay formula, Eq. (1.45). These corrections are therefore also independent from the particular initial state which decays. Furthermore, the terms indicated as $O(\frac{\Lambda^2}{m_b^2})$ in the right-hand-side of Eq. (2.23) can be estimated by resorting to other phenomenological observations in the heavy quark physics.

One problem underlies the previous discussion. For an accurate use of Eq. (2.23), as desirable to get from it $V_{cb}$, a very accurate independent knowledge of $m_b$ is required. For that, another physical observable must be used. For example, in terms of the same $m_b$, the $Y$ mass is also given as

$$M_Y = 2m_b \left[ 1 + O(\alpha_S^2) + O(\frac{\Lambda^2}{m_b^2}) \right].$$

(2.24)

Notice in particular that both perturbative and non-perturbative corrections in the squared parentheses of Eqs. (2.23,24) do depend on the precise definition of $m_b$ itself.

Along these lines, the presently quoted value for $V_{cb}$ is [26]

$$|V_{cb}| = 0.041 \pm 0.004(\text{th}) \pm 0.001(\text{exp}).$$

(2.25)

As said at the beginning of this section, much of the issue is in the assessment of the theoretical error of this determination: the one given in Eq. (2.25) is considered to be a conservative estimate.

Analogous considerations to the ones developed so far can be made in the case of the decays into a final state containing an up-quark instead of a charmed quark, which are controlled by $V_{ub}$ instead of $V_{cb}$. In this case, however, the experiment has only access to the differential decay probability in the high region of the squared invariant mass of the leptons, which is where the decay into a light quark can be kinematically distinguished from the far more abundant decay into a heavier charm quark. In turn, the prediction of the differential decay spectrum is a delicate one, more than for the total decay probability. In any way, from the study of $H_b \rightarrow X_u\bar{v}_l$, one currently quotes [26]
\[ \left| \frac{V_{ub}}{V_{cb}} \right| = 0.008 \pm 0.002. \]  

(2.26)

2.4 Flavour Changing Neutral Currents. \( V_{td}, V_{ts} \)

We have seen in section 2.1, as a main property of the SM, that the neutral current, very much as the electromagnetic one, conserves flavour. On the other hand, one often talks of Flavour Changing Neutral Currents (FCNC). The related phenomenology is another subject that has always been at the centre of the construction of the theory of the weak interactions. For example, it has motivated the introduction of charm in 1970 [20].

To start with, it is useful to give a non ambiguous definition of a FCNC process. By that we mean here a process with a change of quark flavour, which is not predominantly mediated, at short distances, by one W-exchange. That W-exchange(s) be ultimately involved in any flavour changing process, follows again from section 2.1, where we have shown that all flavour violations in the SM reside in the charged current weak interaction Hamiltonian. However, let us think of describing such a process according to the logical steps used in section 2.2 for a typical flavour conserving process like the \( \beta \)-decay of the neutron: first dealing with the bare effective Hamiltonian at short distances, of the order of the Compton wavelength of the W, and then dressing it, at distances of order of 1 GeV or less, with gluons or photons before computing the final matrix element. In the case of a FCNC process, such a calculation, with the bare effective Hamiltonian obtained by one W-exchange, does not give the dominant contribution. The interest of FCNC processes resides in the fact that the short distance Hamiltonian which accounts for the dominant contribution is an electroweak loop diagram, as opposed to a tree level exchange, with relevant internal momenta significantly larger than 1 GeV, say from \( m_b \) to \( M_W \) or, in some cases, \( m_t \). In view of this, it goes without saying that the typical rates of FCNC processes are significantly smaller than for normal weak interaction processes.

If we strictly stick to the definition that we have just given, the observed FCNC processes are very few. A complete list includes CP-violation in K-decays (not discussed in these lectures), the mass differences between \( B_d - \bar{B}_d \) and \( B_s - \bar{B}_s \) and the radiative \( b \rightarrow s \gamma \) decay. We discuss them in the following, including also some consideration of K-decays.

2.4.1) Mixings among the neutral B-mesons
At short distances, the effective Hamiltonian for $\Delta b = 2$, $B - \bar{B}$ mixing arises from a box diagram with two $W$'s and two up-type quark internal propagators. (See Fig. 2.1)

![Box diagram](image)

Figure 2.1: Box diagram contributing to $H_{\text{eff}}^{\Delta b = 2}$.

At any of the four vertices, a CKM matrix element appears connecting the external $Q = -\frac{1}{3}$ quark with the internal $Q = \frac{2}{3}$ quark. With the inclusion of these matrix elements, for any of the internal quark lines in the diagram there is a factor (for an external d-quark)

$$
\sum_{i=u,c,t} V_{ib} S(m_i) V_{id}^* 
$$

(2.27)

where $S(m_i)$ is the quark propagator. Upon use of the unitarity of the CKM matrix, this factor can be rewritten as

$$
V_{tb} V_{td}^* (S(m_t) - S(m_d)) + V_{cb} V_{cd}^* (S(m_c) - S(m_u)).
$$

(2.28)

Notice that pairs of degenerate quarks do not contribute in this factor, as required by the fact that a mixing angle in this case cannot even be defined. This is the so called "GIM cancellation" [20]. In the case under consideration, the balance between the mixing angles and the different quark masses is such that the first term, involving the top quark wins over the second one. The loop integral, dominated by momenta of the order of $M_W + m_t$, easily leads to the effective Hamiltonian

$$
H_{\text{eff}}^{\Delta b = 2} = \frac{G^2 m_t^2}{16 \pi^2} V_{tb} V_{td}^* \int \left( \frac{m_t^2}{M_W^2} \right) \left( \bar{b} \gamma_\mu (1 + \gamma_5) d \right) \left( \bar{b} \gamma_\mu (1 + \gamma_5) d \right) + h.c.
$$

(2.29)
Similarly to what was done in section 2.2, the rescaling of this Hamiltonian from the $M_W + m_t$ scale down to $m_b$ as an effect of gluon corrections introduces an overall factor

$$c(m_b) = \left( \frac{\alpha_s(M_W)}{\alpha_s(m_b)} \right)^{c_6}.$$  \hspace{1cm} (2.30)

(To separate the two scales $M_W$ and $m_t$ would require a RG rescaling between the two, analogous to the one that we make to pass from $M_W$ to low energies. This has been done in Ref. [27] and is numerically not irrelevant). Finally the matrix element of the four-quark operator between B-meson states is required, which is conventionally parametrized as

$$\langle B_d | \bar{b} \gamma_\mu (1 + \gamma_5) d | \bar{b} \gamma_\mu (1 + \gamma_5) d | B_d \rangle = \frac{8}{3} B_B f_B^2 m_B^2$$ \hspace{1cm} (2.31)

where $m_B$ and $f_B$ are the mass and the decay constant of the B-meson respectively and $B_B$ is a numerical fudge factor, estimated by various means to be close to unity. Putting everything together, one finds for the ratio between the mass difference of the two physical B-mesons $m_B \Delta M = 2\langle B_d | H_{\text{eff}}^{\Delta b=2} | B_d \rangle$ and their decay width $\frac{1}{\tau_B}$

$$x_d = \frac{\Delta M}{\Gamma} = 0.55 \frac{G^2 m_t^2}{6\pi^2} |V_{tb} V_{td}^\ast|^2 f \left( \frac{m_t^2}{M_W^2} \right) m_B B_B f_B^2 \tau_B.$$ \hspace{1cm} (2.32)

From the comparison with the experimental result, $x_d = 0.71 \pm 0.06$ [1], obtained by combining several experiments, taking into account the uncertainties in $B_B f_B^2$ and a rather safe range of values for $m_t = 110 \pm 200 GeV$, one gets a first direct determination of $V_{td}$ [26]

$$0.005 \leq |V_{td}| \leq 0.016$$ \hspace{1cm} (2.33)

consistent with the unitarity bound $|V_{td}| \leq 0.018$.

Completely analogous considerations can be made for the case of the $B_s$-meson. Quite clearly, with an obvious meaning of the notation, one has
In view of the unitarity constraint on $V_{ts}$, Eq. (2.34) leads to the expectation of a large value for the mixing parameter $x_5$, between 7 and about 50.

2.4.2) Radiative $b$-decays

In complete analogy with the case of the mixing between neutral B-mesons, the calculation starts from the short distance loop diagram which accounts for the $b \to s \gamma$ single quark transition. (See Fig. 2.2).

![Penguin diagrams contributing to $H_{\gamma}^{b \to s \gamma}$](image_url)

Figure 2.2: Penguin diagrams contributing to $H_{\gamma}^{b \to s \gamma}$

As in the previous case, the integral is dominated by momenta of order $M_W + m_t$. The effective Hamiltonian this time is an off-diagonal magnetic moment operator

$$H_{\gamma}^{b \to s \gamma} = e \frac{Gm_b}{\sqrt{16\pi^2}} V_{tb} V_{ts}^* \sigma_{\mu\nu} \left(\frac{m_s^2}{M_W^2}\right) F_{\mu\nu},$$

where $F_{\mu\nu}$ is the electromagnetic field strength and $g(x)$ is a known function. With respect to the case treated in the previous sub-section, or in section 2.2, a main difference arises because, in dealing with the QCD rescaling of the Hamiltonian (2.35) to lower energies, the magnetic moment operator is not multiplicatively renormalized: in other words, it does not simply involve a rescaling of the short distance piece of the Hamiltonian, but also the inclusion of several other operators able to mediate the $b \to s$ transition [28]. The mixing with these other operators is induced by two loop diagrams.

As a result of this complication, the prediction for the radiative $b$-decay is affected by a non negligible uncertainty. One typically obtains, for the inclusive radiative decay [26]
\[ BR(B \to \chi_s \gamma) = (3.0 \pm 1.2) \cdot 10^{-4} \]  

(2.36)

with about 30\% of the uncertainty accounting for the effect of the top mass variation between 100 and 200 GeV. Notice, on the other hand, that the BR in Eq. (2.36) depends on a combination of CKM matrix elements, \( \left| \frac{V_{tb}V_{ts}}{V_{cb}} \right|^2 \), which is pretty well know, by a combination of experiment and unitarity constraints. Eq. (2.36) is in good agreement with the recent experimental result [29]

\[ BR(B \to \chi_s \gamma)_{\text{exp}} = (2.32 \pm 0.67) \cdot 10^{-4} \]  

(2.37)

2.4.3) K-decays

Of the several rare K-decays observed at present, none deserves the name of FCNC in the sense defined above. For example, \( K^+ \to \pi^+ e^+ e^- \) is likely to have comparable contributions from short distance and long distance physics. Even the very rare \( K_L \to \mu^+ \mu^- \), with a measured rate below \( 10^{-8} \), is dominated by a long distance intermediate state with two photons.

On the other hand, there is one K-decay, \( K^+ \to \pi^+ \nu \bar{\nu} \), which, on one side, is a genuine FCNC process and, on the other side, has a rate that might be measurable in a not too distant future. The short distance one loop diagrams that give rise to the bare effective Hamiltonian relevant to this decay are shown in Fig. 2.3.
Figure 2.3: Box and penguin diagrams contributing to $H_{\text{eff}}^{\nu \nu}$

They include both box and "penguin-type" vertex diagrams. Along similar lines to those developed in the previous subsections, the explicit calculation gives

$$H_{\text{eff}}^{\nu \nu} = \frac{G}{\sqrt{2}} \frac{\alpha}{2 \pi \sin^2 \theta} \sum_{i=c,t} V_{is}^* V_{id} X \left( \frac{m_t^2}{M_W^2} \right) \left( \bar{\nu} \gamma_\mu (1 + \gamma_5) d \right) \left( \bar{\nu} \gamma_\mu (1 + \gamma_5) v \right) + \text{h.c.}, \quad (2.38)$$

where

$$X(x) = \frac{x}{8} \left[ -\frac{2 + x}{1 - x} + \frac{3x - 6}{(1 - x)^2} \ln x \right].$$

To obtain Eq. (2.38), the expression (2.28) for the internal up-type quark line has been used, (with $b$ replaced by $s$), and both terms are kept, since the values of the relevant mixing angles are such that the charm contribution is not completely negligible with respect to the top one. In both cases we have set $m_u = 0$. Let us discuss the probably dominant top contribution, which is also simpler to deal with, because the relevant momentum in the short distance loop is of the order $M_W + m_t$.

To proceed further, at least this contribution to the decay amplitude has two advantages with respect to the previous cases. For reasons that have been discussed in sections 1.11 and 2.2, it does not get any logarithmic corrections either from gluons or from photon exchanges. Furthermore, the relevant hadronic matrix element is related, via an isospin rotation, to the matrix element occurring in the decay $K^+ \rightarrow \pi^0 e^+ \nu$. As a consequence, from the top term in Eq. (2.38) one easily gets

$$\text{BR}(K^+ \rightarrow \pi^0 e^+ \nu) = \frac{\alpha^2 |V_{us}|^2}{2 \pi^2 \sin^4 \theta} X^2 \left( \frac{m_t^2}{M_W^2} \right) \text{BR}(K^+ \rightarrow \pi^0 e^+ \nu). \quad (2.39)$$
Numerically, summing over the 3 neutrino flavours, using Eq. (2.21), $\sin^2 \theta = 0.23$ and [1]

$$BR(K^+ \rightarrow \pi^0 e^+ \nu) = 4.82 \cdot 10^{-2},$$

(2.40)

one obtains (the approximate result being for $m_t = 170 GeV$)

$$\sum_i BR(K^+ \rightarrow \pi^+ v_i \bar{v}_i) = 2.8 \cdot 10^{-11} \left| \frac{V_{ts} V_{td}}{0.04 \cdot 0.01} \right|^2 \times 2 \left( \frac{m_c^2}{M_W^2} \right) \approx 6.7 \cdot 10^{-11} \left| \frac{V_{ts} V_{td}}{0.04 \cdot 0.01} \right|^2.$$  (2.41)

This result has no significant uncertainty, apart from the values of the CKM matrix elements themselves. As mentioned above, however, the charm contribution to the amplitude is probably not negligible. Even for this contribution, of course, it is formally true that there is no large QCD log for going to lower energies, starting from the relevant scale of the short distance lowest order contribution given in Eq.(2.28). The point, however, is that the relevant momentum in the charm contribution goes from $M_W$ to $m_c$, so that, in the same short distance contribution itself, there are corrections of order $\alpha_s \log \frac{M_W}{m_c}$ that need to be resummed. This has been done [30], with the general result that the addition of the charm contribution increases the rate (2.41) by a relative amount that can go up to about 70%, depending in particular on the relative phase of $V_{cs}^* V_{cd}$ with respect to $V_{ts}^* V_{td}$.
3. QUANTUM CORRECTIONS (PRECISION TESTS)

3.1 Definition of the tree level theory

The observed FCNC phenomena discussed in the previous Chapter show evidence for quantum electroweak loops. Both the mass difference in neutral B-mesons and radiative b-decays have their origin in a genuine electroweak radiative correction, not of pure electromagnetic nature. In fact, due to the importance in both cases of the virtual top contribution, they constitute evidence for the existence of the top quark at all. Through a logarithmic sensitivity to $m_t$, they also clearly indicate a heavy top, with a preferred mass between 100 and 200 GeV. However, due to the presence of light quarks in the external states, the FCNC processes have no sensitivity to the Higgs at all, at least in the SM. Their dependence on poorly or not at all known elements of the CKM matrix is a further limitation of the FCNC processes in unravelling the structure of the theory at the quantum level.

In this Chapter a global discussion is given of all the physical observables in electroweak physics which have been measured with significant precision, mostly in $e^+ - e^-$ experiments at the $Z$ resonance, and whose theoretical prediction does not involve any of the CKM matrix elements. Attention is payed to describe the structure of the radiative corrections in a way that allows a comparison of experiment with theory, not restricted to the SM only.

We consider a generic theory that fulfils the following requirements:

i) The gauge group is SU(2)$\times$U(1);

ii) The spectrum includes the standard three generations of fermions with the usual SU(2)$\times$U(1) assignments;

iii) At the tree level, the vector boson masses are related to the gauge couplings and to the order parameter $\nu$ by the relations

$$M_W^2 = \frac{g^2 \nu^2}{2}, \quad M_Z^2 = \frac{(g^2 + g'^2)\nu^2}{2}$$ (3.1)

By the first requirement, one discards the possibility that, at the tree level, the $W$ and the $Z$ bosons be mixed with the gauge bosons of a possibly larger gauge group (e.g., with an extra U(1) factor). As discussed in section (1.9), the third condition is dictated by the experimental observation that the $\rho$-parameter equals one within few per mille. In this way one certainly includes the SM itself [2], a generic multi-Higgs doublet model, the Minimal Supersymmetric Standard Model (MSSM) [31] and QCD-like TechniColour (TC) models [32], all of which give the
same tree level predictions for the precision observables that we shall consider. In an
SU(2)\times U(1) gauge theory with any number of Higgs doublets \( H_i \), like the MSSM
itself where \( i = 1, 2 \), it is always possible to define the normalized linear combination

\[
H = \frac{\sum_i v_i H_i}{\sqrt{\sum_i v_i^2}}, \quad \langle H_i^0 \rangle = v_i
\]  

(3.2)

which gets a vacuum expectation value

\[
v = \sqrt{\sum_i v_i^2}
\]

(3.3)

and plays the same role as the SM Higgs boson in giving rise to the \( W \) and \( Z \) masses
(3.1). In the case of a technicolour theory, with dynamical symmetry breaking
generated by a condensation of a fermion bi-linear, as in QCD, the role of the larger
symmetry of the SM Higgs potential in guarantying the relations (3.1), (See section
1.9), is played by the chiral global symmetry of the QCD-like Lagrangian.

3.2 "Basic observables"

In this way one is led to consider a theory described by a Lagrangian
\( L(g, g', v; ...) \), which predicts, at the tree level, in terms of \( g, g' \) and \( v \) only, a series of
"precision observables". The dots in \( L(g, g', v; ...) \) stand for the many other possible
parameters, e.g. the top or the Higgs masses. The "precision observables", by
definition, can be influenced by these extra parameters only via radiative corrections.

Given this framework, even sticking only to the tree level predictions, three
measurements are needed to determine the basic parameters, whereas any extra
measurement can be used to test the theory (or determine, via radiative corrections,
the remaining parameters). The basic observables that are used to determine the
theory must be precisely measured, on one side, and theoretically calculable in a
clean way on the other side. Three quantities neatly emerge:

i) The electromagnetic fine structure constant, \( \alpha \), as measured by the Josephson
effect or the electron \( g - 2 \)[1]

\[
\alpha = 137.0359895 (61).
\]

(3.4)

ii) The Fermi constant \( G \)[1]

\[
G = 1.16637(2) \times 10^{-5} \text{ GeV}^{-2}
\]

(3.5)
as determined from the muon lifetime $\tau_\mu$ and the theoretical formula (a special case of Eqs. (1.43,45)) [18]

$$
\frac{1}{\tau_\mu} = \frac{G^2 m_\mu^5}{192\pi^2} \left( 1 - \frac{m_e^2}{2 m_\mu} \right) \left[ 1 + \left( \frac{\alpha}{\pi} \right) \frac{25}{4} \pi^2 \left( 1 + \frac{2 \alpha}{3\pi} \frac{m_\mu}{m_e} \right) \right].
$$  \hspace{1cm} (3.6)

iii) The Z-mass [33]

$$
M_Z = 91.1888 \pm 0.0044 \text{GeV}
$$  \hspace{1cm} (3.7)

The relative uncertainty on $M_Z$, which is the less precisely known among the basic observables, is far smaller (by more than one order of magnitude at least) than the relative uncertainty in any of the other observables that we shall consider.

At the tree level, the basic observables are expressed in terms of the parameters $g, g', v$ as (See Chapter 1)

$$
\alpha_0 = \frac{g^2 s^2}{4\pi}, \quad G_0 = \frac{\sqrt{2}}{4v^2}, \quad M_{Z0}^2 = \frac{g^2 v^2}{2c^2}
$$  \hspace{1cm} (3.8)

where$^{(f1)}$

$$
s^2 = \frac{g^2}{g^2 + g'^2} \quad \text{and} \quad c^2 = 1 - s^2.
$$  \hspace{1cm} (3.9)

### 3.3 Derived observables. Tree level values

Eqs. (3.8,9) can be solved in favour of $g, g'$ and $v$ as functions of $\alpha_0, G_0, M_{Z0}$. In this way, at the tree level, the precision observables that are used to test the theory can all be expressed in terms of $\alpha_0, G_0, M_{Z0}$. To this end, one needs:

i) the W-mass

$$
M_{W0}^2 = M_{Z0}^2 c_0^2, \quad c_0^2 = 1 - s_0^2 = \frac{1}{2} \left( 1 + \left( \frac{4\pi \alpha_0}{\sqrt{2} G_0 M_{Z0}^2} \right)^{1/2} \right).
$$  \hspace{1cm} (3.10)

$^{(f1)}$ The reader should be aware of the fact that, although keeping the same symbol, the precise meaning of $s^2$ will change as we progress.
ii) the photon coupling to the fermion $f$ of charge $Q_f$

$$V_\mu (\gamma - f \, \bar{f}) = -\sqrt{4\pi\alpha_0 Q_f} \gamma_\mu;$$

(3.11)

iii) the $Z$ coupling to the fermion $f$ of weak isospin $T_{3f}$ and charge $Q_f$

$$V_\mu (Z - f \, \bar{f}) = -\left( \sqrt{2G_0 M_Z^2} \right) \gamma_\mu \left( g^0_{Yf} - g^0_{Af} \gamma_5 \right);$$

(3.12)

$$g^0_{Yf} = T_{3f} - 2Q_f s^2, \quad g^0_{Af} = T_{3f}$$

(3.13)

iv) the $W$-coupling to the fermion doublet $(f, f')$

$$V_\mu (W - f \, \bar{f}) = -\sqrt{4\pi\alpha_0 \over 2\sqrt{2} s_0} \gamma_\mu (1 - \gamma_5).$$

(3.14)

Of special interest are the $Z$ widths into a pair of fermions $\Gamma (Z \rightarrow f \, \bar{f})$, the forward-backward asymmetries at the $Z$ pole

$$A_{FB}^f = \frac{\sigma_F^f - \sigma_B^f}{\sigma_F^f + \sigma_B^f}$$

(3.15)

$$\sigma_{F(B)}^f = \int \int_0^{-1} d\cos \theta \frac{d\sigma}{d\cos \theta} (e^+e^- \rightarrow f \, \bar{f}),$$

(3.16)

and the $\tau$-polarization asymmetry

$$A_{Pol}^\tau = \frac{\sigma_L^\tau - \sigma_R^\tau}{\sigma_L^\tau + \sigma_R^\tau}$$

(3.17)

$$\sigma_{L(R)}^\tau = \sigma (e^+e^- \rightarrow \tau^\tau (L) \tau^\tau (R)).$$

(3.18)
It is a simple matter to obtain, from these definitions, the following tree-level expressions\(^{(2)}\)

\[
\Gamma^{(0)}(Z \rightarrow f \bar{f}) = N_C \frac{G_0 M_Z^2 Z_0^2}{6\pi \sqrt{2}} \left( s_{Vf}^0 + s_{Af}^0 \right) \tag{3.19}
\]

where \(N_C = 1, 3\) for leptons and quarks respectively, and

\[
A^{\tau(0)}_{Pol} = 2\eta_e \quad A^{f(0)}_{FB} = 3\eta_e \eta_f \tag{3.20}
\]

where

\[
\eta_f = \frac{s_{Vf}^0 s_{Af}^0}{s_{Vf}^0 + s_{Af}^0} \tag{3.21}
\]

If one neglects radiative corrections, it is already possible to obtain the predictions for some of the "derived observables", starting from the input values of the basic observables given in Eqs. (3.4-7). They are compared in Table 1, column A, with the present experimental results.

<table>
<thead>
<tr>
<th>Observable</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Gamma(Z \rightarrow l \bar{l}))</td>
<td>84.99 Mev</td>
<td>83.56 Mev</td>
<td>86.36 Mev</td>
<td>83.96 ± 0.18</td>
</tr>
<tr>
<td>(A^f_{FB})</td>
<td>0.0637</td>
<td>0.0168</td>
<td>0.0351</td>
<td>0.0170 ± 0.0016</td>
</tr>
<tr>
<td>(A^{\tau}_{Pol})</td>
<td>0.296</td>
<td>0.150</td>
<td>0.216</td>
<td>0.143 ± 0.010</td>
</tr>
<tr>
<td>(A^{\bar{b}}_{FB})</td>
<td>0.210</td>
<td>0.105</td>
<td>0.152</td>
<td>0.0967 ± 0.0038</td>
</tr>
<tr>
<td>(M_{w}/M_Z)</td>
<td>0.8876</td>
<td>0.8768</td>
<td>0.8935</td>
<td>0.8798 ± 0.0028</td>
</tr>
</tbody>
</table>

### 3.4 Electromagnetic corrections

The bulk of the radiative corrections is of pure electromagnetic origin. On the other hand, at the level of precision that is of interest here, it is not possible to doubt

\(^{(2)}\) For simplicity Eq. (3.19) does not contain the trivial fermion mass corrections, which are however not completely negligible for the \(\tau\) and the \(b\) quark.
of the effectiveness of QED in describing the radiative corrections from photon exchanges. It is therefore important to consider the pure QED corrections in isolation from the full electroweak corrections, a distinction which is possible always having in mind the required precision.

For $e^+ - e^-$ observables at the Z-pole or for the W mass, the largest effect, by far, from pure QED radiative corrections is the change in the electric charge when going from $q^2 = 0$, where the fine structure constant defined in Eq. (3.4) is measured, to $q^2 = M_Z^2$, which is the relevant momentum scale for the observables in question. This change from $\alpha = \alpha(0)$ to $\alpha(M_Z^2)$ is related to the photon vacuum polarization function

$$\Pi_{\mu\nu}(q) = -ig_{\mu\nu}[A_{\gamma\gamma}(0) + q^2 F_{\gamma\gamma}(q^2)] + q_{\mu}\bar{q}_{\nu} \text{ terms} \quad (3.22)$$

via

$$\alpha(M_Z^2) = \frac{\alpha}{1 + F_{\gamma\gamma}(M_Z^2) - F_{\gamma\gamma}(0)} \equiv \frac{\alpha}{1 - \Delta \alpha} \quad (3.23)$$

In $F_{\gamma\gamma}$ we only include the contributions from the lepton and the light quark loops:

$$\Delta \alpha = \Delta \alpha_l + \Delta \alpha_q \quad (3.24)$$

i.e. we will conventionally include in the remainder of the corrections both the top quark and the W-boson loops. The lepton loops give $\Delta \alpha_l = -0.0314$, whereas, for the quark contribution, a perturbative calculation is not possible, because of strong interaction effects at low $q^2$. The way out consists [34] in relating $\Delta \alpha_q$ to the measured, and properly normalized, hadronic cross-section

$$R(s) = \frac{3s}{4\pi\alpha^2} \sigma(e^+e^- \rightarrow \gamma \rightarrow \text{hadrons}; s) \quad (3.25)$$

via a dispersive representation

$$\Delta \alpha_q = \frac{\alpha}{3\pi} M_Z^2 \int_{4m_W^2}^{\infty} \frac{dsR(s)}{s^2(s - M_Z^2)} \quad (3.26)$$
This gives $\Delta \alpha_q = -0.0282(9)$, with the error dominated by the uncertainties in the measured cross-section between the charm and the beauty threshold. Putting together the lepton and the quark contributions, one obtains

$$\alpha(M_Z)^{-1} = 128.87 \pm 0.12$$

(3.27)

which replaces Eq. (3.4) as the reference value of the fine structure constant. The uncertainty of this new value is not such that one can obviously neglect it in the following. There is in fact at present a debate in the literature on the best value of $\alpha(M_Z)$ that can be extracted from the data on $e^+ - e^+$ into hadrons and on the corresponding uncertainty [35]. We shall come back in section 3.10 on how to deal with values of $\alpha(M_Z)$ possibly different from (3.27).

The other relevant electromagnetic corrections for the $e^+ - e^-$ observables at the Z-pole arise from initial and final-state photon radiation and from the one photon exchange contribution to the scattering amplitudes. Bremstrahlung from the initial particles is in fact the largest source of corrections to the Breit-Wigner shape of the Z resonance. It can be accounted by folding the cross section with a "structure function" $F(x,s)$ that describes the probability that the initial electron or positron emit a photon which carries away a fraction $x$ of the initial momentum:

$$\sigma(s) = \int_0^{1 - s_0/s} dx F(x,s) \sigma_0(s(1 - x))$$

(3.28)

where $s_0$ is a threshold energy below which the final state does not exist or is not detected and $\sigma_0$ is an "uncorrected" cross section that we now define, taking into account of the relativistic phase space and of the one photon exchange amplitude. One has

$$\sigma_0(e^+ e^- \rightarrow f \bar{f}; s) = \sigma^0_{Z, f \bar{f}}(s) + \sigma^0_{\gamma, f \bar{f}}(s) + \sigma^0_{Z\gamma, f \bar{f}}(s)$$

(3.29)

where

$$\sigma^0_{Z, f \bar{f}}(s) = \frac{12 \pi \Gamma_{\gamma f} \Gamma_f}{M_Z^2 \Gamma_{\gamma Z}^2} \frac{s \Gamma_{\gamma Z}^2}{(s - M_Z^2)^2 + s^2 \Gamma_{\gamma Z}^2}$$

(3.30)
and $\sigma_{\gamma, ff}^0(s)$, $\sigma_{Z\gamma, ff}^0(s)$ are respectively the pure photon and the Z-photon interference contributions to the cross section, as computed in the SM. Eq. (3.30) is in fact the very definition of the Z mass and of the Z widths. At the needed level of accuracy, the Z mass so defined coincides with the real part of the pole of the Z propagator.

Finally, photon radiation from the final state $f \bar{f}$ is accounted for by the factor $(1 + 3\alpha/4\pi Q_f)$ multiplying the QED uncorrected width.

The importance of the QED corrections is manifest from column B of Table1. There and hereafter, all the $e^+ - e^-$ observables are defined after the deconvolution of the initial state radiation. As such, their radiative correction effects are largely dominated by the charge renormalization from $q^2 = 0$ to $q^2 = M^2_Z$. The entries in column B are obtained in the same way as in column A, except for the use of (3.27) instead of (3.4) and the introduction of the final state radiation factor in the Z leptonic width.

3.5 Renormalization

To deal with the full electroweak corrections, one has to go through the renormalization procedure. In principle, this opens the way to many possible different renormalization schemes. In a weak coupling theory, however, as the one that we are dealing with, one procedure emerges above all, which avoids useless intermediate steps and speeds up the necessary calculations in a significant way. Technically, it is itself a renormalization scheme, called "on-shell scheme".

First, we focus on the basic observables

$$(\alpha_0, G_0, M_{Z0}) \equiv a_0^i$$

(3.31)

and we compute the radiative corrections to them, in the regularized theory,

$$a_0^i \xrightarrow{RC} a_0^i + \delta a^i(a_0^i) \equiv a^i(a_0^i)$$

(3.32)

In this way we obtain the renormalized basic observables $a^i(a_0^i)$ as power series in $\hbar$, which can be formally inverted to

$$a_0^i = a_0^i(a^i).$$

(3.33)
In the same way we have to consider the loop corrections to the derived observables, or rather to the S-matrix elements that allow to compute the derived observables. As an effect of these corrections, any tree level S-matrix element \( S_0(a^i_0) \) goes into the renormalized one

\[
S_0(a^i_0) \xrightarrow{RC} S_0(a^i_0) + \delta S(a^i_0) \equiv S(a^i_0)
\]

(3.34)

again as a formal power series in \( \hbar \).

One gets the desired connection between the renormalized S-matrix elements, needed to compute the derived observables, and the renormalized basic observables \( a' \), by considering the expansion in \( \hbar \) of \( S(a'_0(a')) \) both in \( S(a'_0) \) and in the \( a'_0(a') \) themselves. It is only at this last step that the coefficients of the series are finite in the limit of infinite cut-off. For example, at one loop, one will have

\[
S(a'_0(a')) = S_0(a'_0) + \delta^{(1)} S(a'_0)
\]

\[
= S_0(a') + \delta^{(1)} S(a') - \sum_i \frac{\partial S_0}{\partial a^i} \delta^{(1)} a^i
\]

(3.35)

which shows explicitly that the finite one-loop shift \( \Delta S^{(1)}(a') \) gets both a direct contribution \( \delta^{(1)} S(a') \) and an indirect one, \(- \sum_i (\partial S_0 / \partial a^i) \delta^{(1)} a'\), from the shifts of the basic parameters, both separately divergent for infinite cut-off. This procedure is followed consistently throughout the following sections.

The fact that the basic observables (3.31) are enough to determine all tree level quantities is what makes it sufficient to renormalize them in order to obtain finite one loop results. To make convergent a two loop calculation requires renormalizing also all the other parameters that intervene at the one loop level. In the case of the SM and for the observables under examination, they are the top mass and the Higgs mass.

### 3.6 Radiative corrections in the "gauge-less" limit. Large \( m_t \) effects

After having examined the electromagnetic effects, with the purpose of dealing with the other corrections in order of importance, we now consider the radiative corrections to the various precision observables that grow like powers of the top-quark mass. From the early work of Veltman [36], it is well known that such effects do arise from top/bottom loop corrections to the W and Z vacuum...
polarization amplitudes, making the $\rho$-parameter deviate from one. This was anticipated in section 1.9. More recently, an analogous effect has been pointed out [37], which leads to the presence of a contribution to the $Z \rightarrow b\bar{b}$ vertex, also growing like a power of $m_t$. Following Ref. [38], one can deal with these effects by means of a Lagrangian that knows nothing about the gauge couplings. The point is that, for a heavy top, the leading corrections both to the $\rho$-parameter and to the $Z \rightarrow b\bar{b}$ vertex can be viewed as power series in the "top fine structure constant" $\alpha_t = g_t^2/4\pi$, where $g_t$ is the top Yukawa coupling. These corrections have clearly nothing to do with the gauge couplings.

With this in mind, let us consider the SM Lagrangian for the third generation of quarks, with the vector bosons treated as external classical currents without kinetic terms, and the bottom Yukawa coupling neglected:

$$L = i \overline{Q}_L D_L Q_L + i \overline{d}_R D_t R + i \overline{b}_R D_b R + \left| D_\mu \phi \right|^2 + V(\phi) + \left( g_t \overline{Q}_R \phi R + h.c. \right) \quad (3.36)$$

where $Q_L = (t, b)_L$. Shifting the Higgs field around the minimum of $V(\phi)$, we set

$$\phi = \left( \begin{array}{c} i\phi^+ \\ v + \frac{1}{\sqrt{2}}(H + i\chi) \end{array} \right), \quad (3.37)$$

where $\chi$ and $\phi^+$ are the Goldstone bosons, eaten by the $Z$ and the $W$ to become massive. In this way we obtain

$$\begin{align*}
L &= \left[ \partial_\mu \phi^+ - i \frac{g_\nu}{\sqrt{2}} W_\mu \right]^2 + \frac{1}{2} \left[ \partial_\mu \chi - i \frac{g_\nu}{\sqrt{2}c} Z_\mu \right]^2 + \frac{1}{2} \left( \partial_\mu H \right)^2 + \\
&\quad + i\bar{t} \partial_t + \left( g_t \bar{t} i \gamma_\mu R + h.c. \right) + i\bar{b} \partial_b + ...
\end{align*} \quad (3.38)$$

The dots in (3.38) stand for all the interactions among the Goldstone bosons, the physical Higgs fields $H$, and the $t, b$ quarks, dependent upon the top Yukawa coupling $g_t$ and the quartic Higgs coupling $\lambda$.

Suppose now that we perform loop corrections with this Lagrangian. We can compactly describe their results in terms of the following effective Lagrangian

$$L^{\text{eff}} = \frac{Z}{2} \left| \partial_\mu \phi^- - i \frac{g_\nu}{\sqrt{2}} W_\mu \right|^2 + \frac{Z_\chi}{2} \left| \partial_\mu \chi - i \frac{g_\nu}{\sqrt{2}c} Z_\mu \right|^2 + \frac{Z_H}{2} \left( \partial_\mu H \right)^2$$

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which contains several (divergent) constants $Z_i = Z_i(g_t, \lambda)$. The important points about Eq. (3.39) are:

i) the presence of the $\chi\bar{b}b$ coupling, which must be of derivative type in the $\chi$ field, because the b-quark is massless (the $b_R$ does not interact);

ii) the fact that the derivative terms in the Goldstone boson fields keep the covariant form in terms of the classical fields $W_\mu$ and $Z_\mu$, as it can explicitly be shown by means of the Ward identities of the SU(2)$\times$U(1) global invariance [38] possessed by the Lagrangian (3.36).

In terms of the constants $Z_i$'s appearing in Eq. (3.39) we can now express the renormalized observables. The gauge couplings do not get any correction, so that, in particular

$$\alpha = \frac{g^2s^2}{4\pi}.$$  \hspace{1cm} (3.40)

The $Z$ and the $W$ masses, from (3.39), become

$$M_Z^2 = Z_2^X \frac{g^2s^2}{2c^2}, \quad M_W^2 = Z_2^\phi \frac{g^2s^2}{2}.$$  \hspace{1cm} (3.41)

The $\mu$-decay, or the Fermi constant, is only affected through the corrections to the $W$ mass, namely

$$G = \frac{\sqrt{2}g^2}{8M_W^2} = \frac{\sqrt{2}}{4v^2Z_2^\phi}.$$  \hspace{1cm} (3.42)

One may also be interested in the neutral-current Fermi constant $G^{NC}$, defined in an analogous way to $G$, for example, from the elastic electron-neutrino amplitude at $q^2 = 0$

$$G^{NC} = \frac{\sqrt{2}g^2}{8M_Z^2c^2} = \frac{\sqrt{2}}{4v^2Z_2^X}.$$  \hspace{1cm} (3.43)
Finally, for the $Z \rightarrow b\bar{b}$ vertex, always from Eq. (3.39) and taking into account the renormalization of the $b_L$ field, one has

$$V_{\mu}^{GIM}(Z \rightarrow b\bar{b}) = \frac{g Z_1 \gamma_{\mu}}{2c Z_2^b} \frac{1 + \gamma_5}{2} = \frac{g}{2c} \gamma_{\mu} \frac{1 + \gamma_5}{2}$$  \hspace{1cm} (3.44)$$

The super-script GIM in $V_{\mu}^{GIM}(Z \rightarrow b\bar{b})$ is there to recall that the top corrections to the vertex are only present in the $b\bar{b}$ case and not for any of the other down-type quarks. As such, this contribution to the $Z \rightarrow b\bar{b}$ vertex is called "GIM violating".

According to the procedure outlined in section 3.3, we have to express all the derived observables in terms of $\alpha$, $G$ and $M_Z$. We have therefore, from Eqs. (3.40-43)

$$G^{NC} = G \frac{Z_o^q}{Z_2^q} = G \rho$$  \hspace{1cm} (3.45)$$

$$s^2 c^2 = \frac{\pi \alpha Z^q}{\sqrt{2}GM_Z^2 Z_o^q} = \frac{\pi \alpha}{\sqrt{2}GM_Z^2 \rho}$$  \hspace{1cm} (3.46)$$

$$M_W^2 = M_Z^2 \rho c^2$$  \hspace{1cm} (3.47)$$

Knowing $\rho \equiv Z_o^q / Z^q$ and $\tau \equiv Z_1 / Z^q$ only, we can in fact express all the radiative effects related to the large top Yukawa coupling. Since, from Eqs. (3.41,42),

$$\frac{g^2}{2c^2} = \sqrt{2}GM_Z^2 \rho,$$  \hspace{1cm} (3.48)$$

for the $Z$ widths into any pair of fermions other than $b\bar{b}$ one has

$$\Gamma(Z \rightarrow f\bar{f} \neq b\bar{b}) = N_C P \frac{GM_Z^2}{6 \pi \sqrt{2}} \left( g_{Vf}^2 + g_{Af}^2 \right)$$  \hspace{1cm} (3.49)$$

where $g_{Vf}$, $g_{Af}$ have their tree level value with $s^2$ given by Eq. (3.46).

On the other hand, for the $Z \rightarrow b\bar{b}$ width, taking into account of the GIM-violating contribution (3.44), one has
Finally, for the asymmetries at the Z-pole, again the tree level expressions hold with $s^2$ given by Eq. (3.46). (The inclusion of $\tau$ in $A^B_{FB}$ according to Eq. (3.51) gives a numerically irrelevant correction due to the smallness of $g_{Vb}$).

After having seen how the large $m_t$ effects spread out, via $\rho$ and $\tau$, in all the precision observables, we can compute $\rho$ and $\tau$ themselves in the SM, using the Lagrangian (3.36). At one-loop level, the only diagrams that contribute to $\rho$ and $\tau$, are shown in Fig.3.1.

From them it is immediate to get the well-known leading-order contributions [36,37]

$$\rho - 1 = 3x, \quad \tau = -2x$$

where we have set

$$x = \frac{Gm_t^2}{8\pi\sqrt{2}},$$

$$\Gamma(Z \to b\bar{b}) = \rho \frac{3GM_Z^2}{6\pi \sqrt{2}} \left( g_{Vb}^2 + g_{Ab}^2 \right)$$

$$g_{Vb} = -\frac{1}{2} \left( \frac{1}{3} - \frac{4}{3} s^2 + \tau \right), \quad g_{Ab} = -\frac{1}{2} (1 + \tau)$$
which, in leading order, coincides with $\alpha_t/8\pi$.

Quite a few more diagrams contribute at the two-loop level. Nevertheless, the simplification achieved by working with the Goldstone Lagrangian rather than the gauge Lagrangian itself makes even the two-loop calculation quite manageable [38]. At this order, also the Higgs quartic coupling $\lambda$ comes in, which can be traded for the Higgs mass $m_h$ by means of the lowest-order relation $m_h^2 = 4\lambda v^2$. Expanding in $x$, the second-order coefficients of both $\rho$ and $\tau$ are functions of $m_H^2/m_t^2$. For $m_H^2/m_t^2 << 1$, they are

$$\rho - 1 = 3x\left(1 + x\left(22 - 2\pi^2\right)\right)$$

(3.54)

$$\tau = -2x\left(1 + x\left(9 - \pi^2/3\right)\right)$$

(3.55)

whereas, for $r = m_t^2/m_H^2 << 1$,

$$\rho - 1 = 3x\left(1 + \frac{x}{4}\left(61 + 4\pi^2 + 54\ln r + 6\ln^2 r\right)\right)$$

(3.56)

$$\tau = -2x\left(1 + \frac{x}{144}\left(311 + 24\pi^2 + 282\ln r + 90\ln^2 r\right)\right)$$

(3.57)

As explained in section 3.3, we have obtained finite second-order coefficients because we have expressed $\alpha_t$ in favour of the renormalized $G$ and $m_t$. It is

$$\alpha_t = \frac{g_t^2}{4\pi} = \frac{Gm_t^2}{\pi\sqrt{2}} \frac{Z_2^0 Z_2^L Z_2^R}{\left(Z_m^t\right)^2}$$

(3.58)

where the various renormalization constants have the usual meaning, as in Eq. (3.39), except that they are computed at the pole of the top propagator, since $m_t$ is defined as the position of the pole itself.

The numerical computation of the asymptotic expressions (3.54-58) as well as of the exact results [38] show that the expansion up to second order is well convergent for all $m_t \leq 300$ GeV, $m_H \leq 2$ TeV. This gives confidence in the use of the perturbative calculation to get an upper bound on $m_t$ from the comparison with the

\[\text{[f4] Eq. (3.54) confirms a previous result obtained in Ref. [40].}\]
experimental data, since the bound that one obtains is well inside this region. If we use, e.g., $m_t=300$ GeV (and any $m_H \lesssim 2$ TeV), where $\rho$ and $\tau$ give the largely dominating corrections, and plug their values in the expressions for any at the observables given above, we obtain in all cases striking deviations from the data (see Table 1, column C). To get an accurate determination of the bound on $m_t$, one actually needs to take into account all the other electroweak corrections that do not grow like powers of $m_t$.

3.7 General one loop expressions

It is time that we discuss the full electroweak radiative correction effects [41-43]. To this purpose, in the following the general expressions are given of the derived observables in terms of the one-loop corrected Green functions, in such a way that one will no longer have to worry about the renormalization procedure. In doing this, the scheme outlined in section 3.3 is closely followed.

The relevant quantities, or the needed ingredients, are:

i) the vacuum polarization amplitudes for the $W$, $Z$ and $\gamma$

$$\Pi_{\mu\nu}^{ij}(q) = -i g_{\mu\nu} \left[ A^{ij}(0) + q^2 F^{ij}(q^2) \right] + q_\mu q_\nu \text{ terms} \quad (3.59)$$

where $i,j = W, \gamma, Z$ or possibly $i,j = 3,0$ for the $W_3$ or the $B$-boson respectively;

ii) the contributions to the vector and the axial form factors at $q^2 = M_Z^2$ in the $Z \to f \bar{f}$ vertex from proper vertex diagrams and fermion self energies only

$$-i e \frac{\bar{v} \gamma_\mu (\delta g_{Vf} - \gamma_5 \delta g_{Af}) \gamma_\nu}{2 s c} \mu \quad (3.60)$$

iii) all the one-loop corrections except the vacuum polarization (boxes, vertices and fermion self-energies) to the $\mu$-decay amplitude at zero external momenta

$$-i G_{V,B} \left[ e \gamma_\mu (1 - \gamma_5) \gamma_\nu (1 - \delta g_{Vf}) \gamma_\nu \right] \mu \quad (3.61)$$

Some comments are in order. In the expression (3.60) for the $Z \to f \bar{f}$ vertex, also a magnetic form factor might be introduced. We assume that the chiral symmetries associated with the various fermions, in the theory under consideration, are controlled by their masses (or their Yukawa couplings). This assumption makes the magnetic form factors negligible to the present purposes. The various functions defined in Eqs. (3.59-61) have in general some imaginary parts. In view of the present phenomenological constraints, we assume that possible further contributions to the
imaginary parts from new relatively light particles give negligible effects. Finally, there is the obvious comment that any of the individual form factors defined in Eqs. (3.59-61) is in general neither finite nor gauge-invariant. It is only when their different contributions are grouped together in the derived observables that finiteness and gauge-invariance will be restored.

In terms of the quantities defined in Eqs. (3.59-61), we can first express the shifts of the basic observables (or the input parameters) as defined in section 3.3 [43]. They are:

\[
\alpha = \alpha_0 + \delta \alpha; \quad \frac{\delta \alpha}{\alpha} = -F \gamma'(0) - 2 \frac{s}{c} \frac{A^Z(0)}{M_Z^2}
\]

(3.62)

\[
M_Z^2 = M_{Z0}^2 + \delta M_Z^2; \quad \delta M_Z^2 = -A_{ZZ}(0) - M_Z^2 F_{ZZ}(M_Z^2)
\]

(3.63)

\[
G = G_0 + \delta G; \quad \frac{\delta G}{G} = \frac{A_{WW}(0)}{M_W^2} + \frac{\delta G_{V.B}}{G}
\]

(3.64)

The derived observables of interest to us are: the Z width into a pair of fermions, \(Z \to f \bar{f}\), the asymmetries at the Z pole and the W mass. At one loop, these observables receive direct contributions from the amplitudes (3.59-61) as well as corrections due to the shifts (3.62-64) of the input parameters. For the W mass one immediately obtains

\[
M_W^2 = M_{Z0}^2 c^2 + \delta M_W^2 - \frac{M_Z^2 c^2}{s^2 - c^2} \left( \frac{2}{2} \frac{\delta \alpha}{\alpha} s^2 \frac{\delta G}{G} - c^2 \frac{\delta M_Z^2}{M_Z^2} \right)
\]

(3.65)

where

\[
s^2 = 1 - c^2 = \frac{1}{2} \left[ 1 - \left( 1 - \frac{4 \pi \alpha}{\sqrt{2 G M_Z^2}} \right)^{1/2} \right]
\]

(3.66)

and

\[
\delta M_W^2 = -A_{WW}(0) - M_W^2 F_{WW}(M_W^2)
\]

(3.67)

To obtain the Z widths and the asymmetries, one first has to write down the \(e^+ e^- \to f \bar{f}\) amplitude close to the Z pole.
\[
A(e^+e^- \rightarrow f \bar{f}) = \sqrt{2} G_0 M_Z^2 \left(1 - F_{ZZ}(M_Z^2) - M_Z^2 F_{ZZ}(M_Z^2)\right) \frac{1}{q^2 - M_Z^2} 
\times \bar{\nu}_e \gamma_{\mu} \left(\frac{1}{2} + 2 \delta g^2 + 2 s c F_{Z \gamma}(M_Z^2) + 2 \delta g_{\gamma} \right) - \gamma_5 \left(\frac{1}{2} + 2 \delta g_{A}\right) \mu_e 
\times \bar{\nu}_f \gamma_{\mu} \left(T_{3f} - 2 s \delta Q_f - 2 s c Q_f F_{Z \gamma}(M_Z^2) + 2 \delta g_{F \gamma} \right) - \gamma_5 \left(T_{3f} + 2 \delta g_{A f}\right) \mu_f
\] (3.68)

from which, using the shifts (3.59-61), one gets

\[
\Gamma(Z \rightarrow f \bar{f}) = \frac{G M_Z^2}{6 \pi \sqrt{2}} N_C \left(1 - \frac{\delta G}{G} - \frac{\delta M_Z^2}{M_Z^2} - F_{ZZ}(M_Z^2) - F_{ZZ}(M_Z^2)\right) 
\times \left(\left(g_{Vf} + \Delta g_{Vf}\right)^2 + \left(g_{Af} + \Delta g_{Af}\right)^2\right).
\] (3.69)

\[
A_{FB}^l = 3 \frac{(g_{Vl} + \Delta g_{Vl})^2 (g_{Al} + \Delta g_{Al})^2}{\left[(g_{Vl} + \Delta g_{Vl})^2 + (g_{Al} + \Delta g_{Al})^2\right]^2}
\] (3.70)

where

\[
g_{Vf} = T_{3f} - 2 s^2 Q_f, \quad g_{Af} = T_{3f}
\] (3.71)

\[
\Delta g_{Vf} = 2 Q_f \delta s^2 + 2 \delta g_{Vf} - 2 s c Q_f F_{Z \gamma}(M_Z^2), \quad \Delta g_{Af} = \delta g_{Af}
\] (3.72)

\[
\delta s^2 = \frac{s^2 c^2}{c^2 - s^2} \left(\frac{\delta G}{G} + \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta \alpha}{\alpha}\right)
\] (3.73)

Both in the amplitude (3.68) and in the forward-backward asymmetry (3.70) we have neglected possible box diagram contributions, which is legitimate for observables at the Z pole. Analogous expressions hold for \(A_{Pol}^l\) and \(A_{FB}^b\), with the lepton couplings \(g_{Vl}, g_{Al}\) replaced by \(g_{Vl} + \Delta g_{Vl}\) and \(g_{Al} + \Delta g_{Al}\), as in (3.70). We have already mentioned that the forward-backward b-asymmetry is practically insensitive to (reasonable) corrections to the \(Z - b \bar{b}\) couplings. For them one can then simply take the tree level expressions with \(s^2\) given in Eq. (3.66).
It is customary, and useful for later purposes, to define three auxiliary dimensionless parameters $\Delta r_w, \Delta \rho, \Delta k'$ which are in direct correspondence with three derived observables $M_w, \Gamma(Z \rightarrow l^+l^-)$ and $A_{FB}^l$ via \([44,45]\)

\[
\left(1 - \frac{M_W^2}{M_Z^2}\right) \frac{M_W^2}{M_Z^2} \equiv \frac{\pi \alpha}{\sqrt{2} M_Z^3 (1 - \Delta r_W)}
\] (3.74)

\[
\Gamma(Z \rightarrow l^+l^-) = \frac{G M_Z^3}{6 \pi \sqrt{2}} \left( g_V^2 + g_A^2 \right) \left( 1 + \frac{3}{4} \alpha \right)
\] (3.75)

\[
A_{FB}^l = \frac{3 g_V^2 g_A^2}{\left( g_V^2 + g_A^2 \right)^2}
\] (3.76)

where

\[
g_A = -\frac{1}{2} \left( 1 + \frac{\Delta \rho}{2} \right), \quad g_V = -4s^2 (1 + \Delta k')
\] (3.77)

and $s^2$ given in Eq. (3.66). Notice that $g_v$ and $g_A$ defined in (3.75,76) have only an auxiliary role, being related to $\Delta \rho, \Delta k'$ through (3.77). By looking at Eqs. (3.49) and (3.75-77), notice also the relation between $\Delta \rho$ and the $\rho$ parameter defined in section 3.6, $\rho \approx 1 + \Delta \rho$. In $\Gamma(Z \rightarrow l^+l^-)$ we have inserted the final state radiation factor. From these definitions one obtains \([46]\)

\[
\Delta r_w = \frac{\delta G}{G} + \frac{c^2}{s^2} \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta \alpha}{\alpha} + \frac{s^2 - c^2}{c^2} \frac{\delta M_W^2}{M_W^2}
\] (3.78)

\[
\Delta \rho = -\frac{\delta G}{G} - \frac{\delta M_Z^2}{M_Z^2} - F_{ZZ} \left( M_Z^2 \right) - M_Z^2 F_{ZZ} \left( M_Z^2 \right) - 4 \Delta g_{Al}
\] (3.79)

\[
\Delta k = \frac{1}{2s^2} \left( \Delta g_{Vl} - (1 - 4s^2) \Delta g_{Al} \right)
\] (3.80)

This is as much as one can do in general, without specifying the actual theory that one wants to deal with: the SM or something else.

3.8 Radiative corrections for infinite Higgs mass in the SM
The infinite Higgs mass limit of the SM is a particularly interesting one [47]. In the radiative corrections, it results in the appearance of divergences, cut-off by the Higgs mass itself, affecting the physical observables. At one and two loops, it is possible to characterise and calculate all the leading divergences in a simple way. Other than those controlled by the Yukawa coupling of the top quark, appearing only at two loops and calculated in section 3.6, they are all contained in the vacuum polarization amplitudes of the vector bosons. In particular, simple power counting arguments show that they can only affect the masses and the kinetic terms of the vector bosons and not the higher derivatives of the vector-boson vacuum-polarization amplitudes. As shown below, this means that the leading divergent terms in the large Higgs mass limit affect any observable via three combinations, at most, of the vacuum polarization amplitudes.

To this purpose, let us consider the vector boson mass terms in the effective Lagrangian given in Eq. (3.39) and let us add to them the general kinetic terms consistent with electric charge conservation

\[ L_{\text{eff}}^{\text{VB}} = -\frac{1}{2} W_{\mu \nu}^+ W_{\mu \nu}^- - \frac{1}{4} B W_{\mu \nu}^3 W_{\mu \nu}^- - \frac{1}{2} C \frac{s}{c} B_{\mu \nu} W_{\mu \nu}^- - \frac{1}{4} D B_{\mu \nu} B_{\mu \nu} \]

\[ \frac{1}{2} Z^\phi g^2 \nu^2 W_\mu^+ W_\mu^- - \frac{1}{4} Z^\chi g^2 \nu^2 (c W_\mu^3 - s B_\mu)^2 \]

where A,B,C,D are renormalization constants, as \( Z^\phi \) and \( Z^\chi \), all related to the VB vacuum polarization amplitudes defined in the previous section. Not all of these constant have physical meaning, however. By the rescalings

\[ W_\mu^{\pm,3} \rightarrow \frac{W_\mu^{\pm,3}}{\sqrt{A}} \quad B_\mu \rightarrow \frac{B_\mu}{\sqrt{D}} \quad g \rightarrow \sqrt{A} g \quad g' \rightarrow \sqrt{D} g' \]

the fermion couplings to the VB are left unchanged, whereas the above Lagrangian goes into

\[ L_{\text{eff}}^{\text{VB}} = -\frac{1}{2} W_{\mu \nu}^+ W_{\mu \nu}^- - \frac{1}{4} B W_{\mu \nu}^3 W_{\mu \nu}^- - \frac{1}{2} C \frac{s}{c} B_{\mu \nu} W_{\mu \nu}^- - \frac{1}{4} D B_{\mu \nu} B_{\mu \nu} \]

\[ -c^2 M_2^2 \frac{Z^\phi}{Z_2^\chi} W_\mu^+ W_\mu^- - \frac{1}{2} M_2^2 (c W_\mu^3 - s B_\mu)^2 \]
with $M_Z^2$ defined in Eq. (3.41) and $s,c$ still keeping the usual relation with the gauge couplings in the fermionic weak currents. This shows that physical observables are determined, other than by $g,g'$ and $M_Z^2$, by only three combinations of the various renormalization constants

$$
\frac{Z_f^0}{Z_2^0} - 1 = \rho - 1 = e_1, \quad 1 - \frac{B}{A} = e_2, \quad \frac{C}{A} = e_3,
$$

which are related, at one loop, to the VB vacuum polarization amplitudes defined in Eqs. (3.59) by

$$
e_1 = \frac{A_{33}(0) - A_{WW}(0)}{M_W^2}, \quad e_2 = F_{WW}(M_W^2) - F_{33}(M_Z^2), \quad e_3 = \frac{C}{s} F_{30}(M_Z^2).
$$

Notice that, to deviate from zero, both $e_1$ and $e_2$ require a breaking of the SU(2) symmetry that is responsible, at the tree level, for $\rho = 1$. (See section 1.9). Therefore, apart from terms proportional to the Yukawa coupling of the top quark, already computed in section 3.6, a non vanishing contribution to them must involve the exchange of a B-boson, since only the hypercharge coupling breaks the "custodial" symmetry. Furthermore, whereas $e_2$ is the ratio of two VB wave-function renormalization constants, $e_1$ can be viewed as the ratio of the wave-function renormalization constants of the charged and neutral Goldstone bosons, as shown again in section 3.6. When considering loops not involving the third generation fermions, all this means that $e_1$ starts at order $g'^2$, whereas $e_2$ is only non vanishing at two loop order with a $g^2 g'^2$ term. (Remember that the B-boson has no direct coupling to the W-bosons).

The one loop diagrams that contribute with a divergence to $e_1$ and $e_3$ for an infinitely heavy Higgs, namely when the Higgs is never excited, are shown in Fig 3.2. In terms of an ultraviolet and infrared cut-off, $\Lambda$ and $\mu$ respectively, the diagrams of Figs. 3.2, both computed at $q^2 = 0$, immediately give [48]

$$
e_1 = -\frac{3\alpha}{8\pi^2} \frac{\Lambda}{\mu} = -1.2 \times 10^{-3} \frac{\Lambda}{\mu}
$$

(3.88a)
The result for $e_1$ is actually obtained by taking the B-boson propagator in the Landau gauge. This is the gauge that has to be used if one wants to compute $e_1$ in terms of Goldstone boson properties only; the corresponding gauge-fixing term, although breaking the local symmetry, respects in fact the global symmetry that is responsible for the existence of the massless Goldstone bosons in the first place. In the complete theory, diagrams with internal Higgs boson lines make the pure Goldstone diagrams of Fig. 3.2 ultraviolet-convergent and replace the ultraviolet cut-off by the Higgs mass [49]. Furthermore, the infrared cut-off in a complete calculation is replaced by the gauge-boson masses.

Similar considerations along these lines allow to compute in a simple way, in terms of a few two loop diagrams, also the leading contribution growing like the Higgs mass squared to all the physical observables [50]. As the one loop logarithmic divergences, they only occur in $e_1$ and $e_3$.

![Figure 3.2: One-loop contributions to $e_1$ and $e_3$ in the SM without Higgs boson internal lines in the approximation discussed in the text. The B-boson propagator is in the Landau gauge.](image)

3.9 Model independent analysis: the $\varepsilon$ - parameters

In this section, a general strategy is defined for the analysis of the electroweak precision tests with the purpose of isolating the interesting effects in the radiative corrections [51]. The aim is to compare the theory with the full set of experimental results given in Table 2. The analysis is not restricted to the SM, but rather it treats the SM as a particularly relevant example. It is based on four dimensionless parameters, $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and $\varepsilon_b$, which represent an efficient parametrization of the small deviations from what is solidly established. Indeed the epsilons are defined in such a way that they are exactly zero in the limit of neglecting all pure weak loop-corrections to a few especially relevant observables (i.e. when only the predictions from the tree level SM plus pure QED and pure QCD corrections are taken into account). This very simple version of improved Born approximation - hereafter
simply called Born approximation - is a good first approximation [52], according to the data. Furthermore, the epsilons are defined in such a way as to be in one to one correspondence, in a sense to be made precise, with the quantities \( \epsilon_1, \epsilon_2, \epsilon_3 \) and \( \tau \) defined in Eqs. (3.83-87) and (3.44) respectively.

Table 2. Experimental results considered in the text

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_Z(GeV) )</td>
<td>91.1888±0.0044</td>
</tr>
<tr>
<td>( \Gamma_T(MeV) )</td>
<td>2497.4±3.8</td>
</tr>
<tr>
<td>( R = \Gamma_h / \Gamma_I )</td>
<td>20.795±0.040</td>
</tr>
<tr>
<td>( \sigma_h = 12 \pi \Gamma_I \Gamma_h / M_Z^2 \Gamma_T^2(nb) )</td>
<td>41.49±0.12</td>
</tr>
<tr>
<td>( \Gamma_I(MeV) )</td>
<td>83.96±0.18</td>
</tr>
<tr>
<td>( \Gamma_h(MeV) )</td>
<td>1745.9±4.0</td>
</tr>
<tr>
<td>( \Gamma_b(MeV) )</td>
<td>382.7±3.1</td>
</tr>
<tr>
<td>( R_{bh} = \Gamma_b / \Gamma_h )</td>
<td>0.2192±0.0018</td>
</tr>
<tr>
<td>( A_{FB}^b )</td>
<td>0.0170±0.0006</td>
</tr>
<tr>
<td>( A_{pol} )</td>
<td>0.143±0.010</td>
</tr>
<tr>
<td>( A_\epsilon )</td>
<td>0.135±0.011</td>
</tr>
<tr>
<td>( A_{FB}^a )</td>
<td>0.0967±0.0038</td>
</tr>
<tr>
<td>( g_V / g_A ) (all asymm–LEP)</td>
<td>0.0716±0.0020</td>
</tr>
<tr>
<td>( A_{LR}(SLD) )</td>
<td>0.1637±0.0075</td>
</tr>
<tr>
<td>( g_V / g_A ) (all asymm–LEP+SLD)</td>
<td>0.0738±0.0018</td>
</tr>
<tr>
<td>( M_W / M_Z(UA2+CDF+D0) )</td>
<td>0.8798±0.0020</td>
</tr>
<tr>
<td>( \alpha_s(M_Z) )</td>
<td>0.118±0.007</td>
</tr>
</tbody>
</table>

The definition of the epsilons is as follows. For \( \epsilon_1, \epsilon_2 \) and \( \epsilon_3 \), we introduce the following linear combinations of \( \Delta \rho, \Delta k' \) and \( \Delta r_w \), defined in the previous section [45,53]:

\[
\begin{align*}
\epsilon_1 &= \Delta \rho, \\
\epsilon_2 &= c^2 \Delta \rho + \frac{s^2 \Delta r_W}{c^2 - s^2} - 2 s^2 \Delta k', \\
\epsilon_3 &= c^2 \Delta \rho + \left( c^2 - s^2 \right) \Delta k' 
\end{align*}
\] (3.89)

We further define \( \epsilon_b \) from \( \Gamma_b \), the inclusive partial width for \( Z \rightarrow b \bar{b} \), according to the relation [51]

\[
\Gamma_b = \frac{GM_Z^3}{2\pi \sqrt{2}} \beta \left( \frac{3 - \beta^2}{2} g_V^2 + \beta^2 g_A^2 \right) R_{QCD} \left( 1 + \frac{\alpha}{12\pi} \right) 
\] (3.90)
where $\beta = \sqrt{1-4m_b^2/m_Z^2}$, with $m_b=4.8$ GeV, $R_{QCD}$ is the QCD correction factor given by

$$R_{QCD} = 1 + 1.2a - 1.1a^2 - 13a^3; \quad a = \frac{\alpha_s(m_Z)}{\pi}$$  \hfill (3.91)

(for $\alpha_s(M_Z) = 0.118$, $R_{QCD} = 1.0428$) and $g_{vb}, g_{Ab}$ are specified as follows

$$g_{Ab} = -\frac{1}{2}\left(1 + \frac{\Delta \rho}{2}\right)(1 + \varepsilon_b)$$  \hfill (3.92)

$$g_{Vb} = \frac{1 - 4/3(1 + \Delta k)s^2 + \varepsilon_b}{1 + \varepsilon_b}$$

The physical meaning of the $\varepsilon_i$ can be understood by looking at their explicit expressions in terms of the amplitudes defined in section 3.7. After linearization, one obtains

$$\varepsilon_1 = \varepsilon_1 - \delta G_{V,B} \frac{G}{3} - 4\delta g_{Al}$$  \hfill (3.93a)

$$\varepsilon_2 = \varepsilon_2 + \frac{2}{3}\left(1 - \frac{4}{3}s^2\right)\delta g_{Ad} + \varepsilon_3 + \frac{2}{3}\left(1 - \frac{4}{3}s^2\right)\delta g_{Vl}$$

The quantities $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and $\varepsilon_b$ are defined in terms of the physical observables $\frac{M_W}{M_Z}, \Gamma(Z \rightarrow l^+l^-), A_{FB}^\gamma$ and $\Gamma_b$. The relations between the basic observables and the epsilons can be linearized, leading to the formulae

$$\varepsilon_4 = F_{\gamma\gamma}(0) - F_{\gamma\gamma}(M_Z^2)$$

$$\varepsilon_5 = M_Z^2 \left( \frac{dq}{d\varepsilon} \right)_{\varepsilon=0}$$  \hfill (3.94)
\[
\frac{M_{\tilde{W}}^2}{M_{\tilde{Z}}^2} = \frac{M_{\tilde{W}}^2}{M_{\tilde{Z}}^2_B} \left(1 + 1.43\varepsilon_1 - 1.00\varepsilon_2 - 0.86\varepsilon_3 \right)
\]  
\[\Gamma_l |_{B} = \Gamma_l |_{B} \left(1 + 1.20\varepsilon_1 - 0.26\varepsilon_3 \right)\]  
\[A_{FB}^\mu |_{B} = A_{FB}^\mu |_{B} \left(1 + 34.72\varepsilon_1 - 45.15\varepsilon_3 \right)\]  
\[\Gamma_b |_{B} = \Gamma_b |_{B} \left(1 + 1.42\varepsilon_1 - 0.54\varepsilon_3 + 2.29\varepsilon_b \right)\]

(3.95a)  
(3.95b)  
(3.95c)  
(3.95d)

The Born approximations, as defined above, of the corresponding quantities on the right hand side of Eqs. (3.95) depend on \(\alpha_s(M_Z)\) and also on \(\alpha(M_Z)\). Defining

\[
\delta a_s = \frac{1}{\pi} [\alpha_s(M_Z) - 0.118] \quad \delta \alpha = \frac{1}{\pi \alpha} \left[ \alpha(M_Z) - \frac{1}{128.87} \right]
\]

we have

\[
\frac{m_{\tilde{W}}^2}{m_{\tilde{Z}}^2 |_{B}} = 0.76883[1 - 0.40\delta \alpha]\]

(3.97a)  
\[\Gamma_l |_{B} = 83.56[1 - 0.19\delta \alpha] MeV\]

(3.97b)  
\[A_{FB}^\mu |_{B} = 0.01683(1 - 34\delta \alpha)\]

(3.97c)  
\[\Gamma_b |_{B} = 379.6[1 + 1.0\delta a_s - 0.42\delta \alpha] MeV\]

(3.97d)
Figure 3.3: The quantities $\epsilon_1$, $\epsilon_2$, $\epsilon_3$, $\epsilon_b$ as predicted in the SM. The bands correspond to the dependence on $m_H = 50 + 1000 GeV$

It is an important property of the epsilons that, in the SM, for all observables at the $Z$ pole, the whole dependence on $m_t$ (and $m_H$) arising from one-loop diagrams only enters through the epsilons. The same is actually true, at the relevant level of precision, for all higher order $m_t$-dependent corrections, which enter through the vacuum polarization and the $Z \rightarrow b\bar{b}$ vertex. Actually, the only residual $m_t$ dependence of the various observables not included in the epsilons is in the terms of order $\alpha_s^2$ in the pure QCD correction factors to the hadronic widths [54]. But this one is quantitatively irrelevant, especially in view of the errors connected to the uncertainty on the value of $\alpha_s$ itself. The theoretical values of the epsilons in the SM are given in Table 3 and shown in Fig. 3.3. It is important to remark that the theoretical values of the epsilons in the SM, as defined in Eqs. (3.89-93) or, equivalently, in Eqs. (3.95-97) are not affected, at the percent level or more, by reasonable variations of $\alpha_s(M_Z)$ and/or $\alpha(M_Z)$ around their central values. By our definitions, in fact, no term of order $\alpha_s^n$ or $\alpha \ln \left( \frac{M_Z}{m} \right)$, with $m$ a light fermion mass, contributes to the epsilons.
Table 3: Values of the epsilons in the Standard Model as functions of $m_t$ and $m_H$ as obtained from recent versions of ZFITTER [55] and TOPAZ0 [56].

<table>
<thead>
<tr>
<th>$m_t$ (GeV)</th>
<th>$m_H$ (GeV)</th>
<th>$\varepsilon_1$</th>
<th>$\varepsilon_2$</th>
<th>$\varepsilon_3$</th>
<th>$\varepsilon_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>65</td>
<td>300</td>
<td>1000</td>
<td>65</td>
<td>300</td>
<td>1000</td>
</tr>
<tr>
<td>120</td>
<td>1.51</td>
<td>0.888</td>
<td>-0.23</td>
<td>-5.72</td>
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<tr>
<td>130</td>
<td>2.19</td>
<td>1.54</td>
<td>0.413</td>
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<td>-5.74</td>
</tr>
<tr>
<td>140</td>
<td>2.93</td>
<td>2.25</td>
<td>1.10</td>
<td>-6.46</td>
<td>-6.07</td>
</tr>
<tr>
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<td>3.00</td>
<td>1.84</td>
<td>-6.80</td>
<td>-6.38</td>
</tr>
<tr>
<td>160</td>
<td>4.56</td>
<td>3.81</td>
<td>2.63</td>
<td>-7.13</td>
<td>-6.70</td>
</tr>
<tr>
<td>170</td>
<td>5.47</td>
<td>4.68</td>
<td>3.47</td>
<td>-7.48</td>
<td>-7.03</td>
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<td>180</td>
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<td>5.60</td>
<td>4.36</td>
<td>-7.84</td>
<td>-7.36</td>
</tr>
<tr>
<td>200</td>
<td>8.53</td>
<td>7.6</td>
<td>6.27</td>
<td>-8.64</td>
<td>-8.08</td>
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<tr>
<td>210</td>
<td>9.67</td>
<td>8.69</td>
<td>7.30</td>
<td>9.08</td>
<td>-8.47</td>
</tr>
<tr>
<td>230</td>
<td>12.2</td>
<td>11.0</td>
<td>9.49</td>
<td>-10.0</td>
<td>-9.36</td>
</tr>
</tbody>
</table>

In terms of the epsilons, the following expressions hold, within the SM, for the various precision observables

\[
\Gamma_T = \Gamma_T^0 (1 + 1.35\varepsilon_1 - 0.46\varepsilon_3 + 0.35\varepsilon_b)
\]
\[
R = R_0^0 (1 + 0.28\varepsilon_1 - 0.36\varepsilon_3 + 0.50\varepsilon_b)
\]
\[
\alpha_h = \sigma_h^0 (1 - 0.03\varepsilon_1 + 0.04\varepsilon_3 - 0.20\varepsilon_b)
\]
\[
x = x^0 (1 + 17.6\varepsilon_1 - 22.9\varepsilon_3)
\]
\[
R_{bh} = R_{bh}^0 (1 - 0.06\varepsilon_1 + 0.07\varepsilon_3 + 1.79\varepsilon_b)
\]

where $x = \frac{gV}{g_A}$ as obtained from $A_{FB}^\mu$. The quantities in Eqs. (3.95) and (3.98) are clearly not independent and the redundant information is reported for convenience. By comparison with the code of Ref. [55] (the results are also checked with the programme of Ref. [56]) one obtains

\[
\Gamma_T^0 = 2488.9 (1 + 0.73\delta\alpha_S - 0.35\delta\alpha) \text{MeV}
\]
\[
R_0^0 = 20.818 (1 + 1.05\delta\alpha_S - 0.28\delta\alpha)
\]
\[
\sigma_h^0 = 41.422 (1 - 0.41\delta\alpha_S + 0.03\delta\alpha) \text{nb}
\]
\[
x^0 = 0.07531 - 1.32\delta\alpha
\]
\[
R_{bh}^0 = 0.21823
\]
Note that the quantities in Eqs. (3.99) should not be confused, at least in principle, with the corresponding Born approximations, due to small "non universal" electroweak corrections. In practice, at the relevant level of approximation, the difference between the two corresponding quantities is in any case significantly smaller than the present experimental error, from a factor of 2 in the case of $\Gamma_Z$ up to a factor of 6 in $R_{bh}$.

The properties of the epsilons, as precisely defined from Eqs. (3.95-97), make them suitable for a model independent analysis of the electroweak precision tests. In particular, the fact that, for all observables at the $Z$ pole, the whole relevant dependence on $m_t$ (and $m_H$) only enters through the epsilons, is true for any extension of the SM with the property that all possible deviations only occur through vacuum polarisation diagrams and/or the $Z\rightarrow b\bar{b}$ vertex. In any such model, of course, the actual values of the epsilons will differ in general from the SM ones. For this kind of models, however, one can compare the theoretical predictions with the experimental determination of the epsilons as obtained from the whole set of $e^+ - e^-$ high energy data. If a particular model does not satisfy this requirement, then the comparison is to be made with the epsilons determined from the defining variables only, Eqs. (3.95-97), or with some more limited enlargement of the same set of data, depending on the particular case. For example, if lepton universality is maintained, then the data on $A_{FB}^1$ can be replaced by the combined result on $g_V/g_A$ from all lepton asymmetries.

In principle, any four observables could have been picked up as defining variables. In practice we choose those that have a more clear physical significance and are more effective in the determination of the epsilons. In fact, since $\Gamma_b$ is actually measured by $R_{bh}$ (which is nearly insensitive to $\alpha_S$), it is preferable to use directly $R_{bh}$ itself as defining variable, as we shall do hereafter. In practice, since $R_{bh0}$, Eq. (3.99e), is numerically indistinguishable from the Born approximation of $R_{bh}$, this determines no change in any of the equations given above, but simply requires the replacement of Eqs. (3.95d,97d) with Eqs. (3.98e,99e) among the defining relations of the epsilons. In this way, the equations that have completely general validity are (3.95a,b,c and 3.98e), togheter with (3.97a,b,c) and (3.99e), whereas the remaining observables and the corresponding equations, among which (3.95d, 3.97d), can be included in the analysis only according to the progression of hypotheses that we shall discuss.

3.10 Comparison with experiment
By combining the value of $\frac{M_W}{M_Z}$ with the LEP results on the charged lepton partial width and the forward-backward asymmetry, all given in Table 2, one obtains from Eqs. (3.95a,b,c) and (3.97a,b,c):

$$\epsilon_1 = (4.7 \pm 2.2) \times 10^{-3}$$
$$\epsilon_2 = (-3.2 \pm 5.0) \times 10^{-3} + 0.23\delta\alpha$$
$$\epsilon_3 = (3.4 \pm 3.0) \times 10^{-3} - 0.77\delta\alpha$$

Finally, by adding the value of $R_{bh}$ and using Eqs. (3.98e,99e) one finds:

$$\epsilon_b = (2.5 \pm 4.6) \times 10^{-3}$$

The central values of the epsilons, as determined experimentally, depend on the chosen value of $\alpha(M_Z)$, since the Born approximation of the defining variables does. As before, we have taken $\alpha(M_Z) = 1/128.87$ [34] but, in Eqs. (3.100,101), we have given the variation induced on the epsilons by corresponding shifts of $\alpha(M_Z)$. As mentioned in section 3.4, there is a lively debate in the literature on the best value of $\alpha(M_Z)$ that can be extracted from the data on $e^+e^-\rightarrow$ hadrons and on the corresponding uncertainty [35]. By using Eqs. (3.100,101) the reader can easily adapt the results to his/her preferred values.
Defining Variables

In Fig. 3.4 the experimental 1σ ellipse in the $\epsilon_1 - \epsilon_3$ plane is shown and compared, as a particularly relevant example, with the SM predictions for different $m_t$ and $m_H$ values. We recall that $\epsilon_1$ and $\epsilon_3$ are completely determined by $\Gamma_l$ and $A_{FB}^l$. Unlike $\epsilon_1$ and $\epsilon_3$, $\epsilon_2$ and $\epsilon_b$ do not show yet any deviation from zero. In the case of $\epsilon_2$, there is consistency with the SM prediction at all practical values of $m_t$. (See Fig. 3.3). Note that $\epsilon_2$ also depends on $\frac{M_W}{M_Z}$ and better measurements of this quantity are needed in order to make this test more stringent. On the contrary, $\epsilon_b$ would prefer relatively small values of $m_t$. (See Fig. 3.3) This result is a simple and direct consequence of the fact that the measured value of $R_{bh}$ is a bit high (for $m_t \sim 170$ GeV, $\Gamma_b$ is about 2σ larger than the SM prediction).

To proceed further, and include other measured observables in the analysis we need to make some dynamical assumptions. The minimum amount of model dependence is introduced by including other purely leptonic quantities at the $Z$ pole.
such as $A_{pol}^\tau$, $A_e$ (measured [33] from the angular dependence of the $\tau$ polarisation) and $A_{LR}$ (measured by SLD [57]). At this stage, one is simply relying on lepton universality. With essentially the same assumptions one can also include the data on the b-quark forward backward asymmetry $A_{FB}^b$. In fact it turns out that $A_{FB}^b$ is almost unaffected by the $Z\rightarrow b\bar{b}$ vertex correction.

As a result, we can combine the values of $x = g_V/g_A$ from the whole set of asymmetries measured at LEP (obtaining the value given in Table 2) and we can include, in the fit of the epsilons, Eqs. (3.98d,99d), valid in a more general theory fulfilling the stated assumptions. At this stage, with the SLD result also taken into account, the best values of $\epsilon_1, \epsilon_2$ and $\epsilon_3$ are modified according to

$$
\begin{align*}
\epsilon_1 &= (5.1 \pm 2.2) \times 10^{-3} \\
\epsilon_2 &= (-4.1 \pm 4.8) \times 10^{-3} \\
\epsilon_3 &= (5.1 \pm 2.0) \times 10^{-3} \\
\epsilon_b &= (2.4 \pm 4.6) \times 10^{-3}
\end{align*}
$$

with a similar dependence on $\alpha(M_Z)$ as in Eqs. (3.100,101). In Fig. 3.5 we report the two ellipses in the $\epsilon_1 - \epsilon_3$ plane that correspond to the data with and without $A_{LR}$ from SLD.
Figure 3.5: The 1σ ellipse in the plane $\varepsilon_1 - \varepsilon_3$ obtained from the data on $\Gamma_l$ and $\frac{gV}{g_A}$ derived from all asymmetries (Table 2), with and without SLD.

All observables measured on the Z peak at LEP can be included in the analysis provided that we assume that all deviations from the SM are only contained in vacuum polarisation diagrams (without demanding a truncation of the $q^2$ dependence of the corresponding functions) and/or the $Z\to b\bar{b}$ vertex. For a global fit of all high energy data we consider $M_W$, $M_Z$, $\Gamma_{Z, h}$, $\sigma_{h, bh}$ and $x=gV/g_A$ given in Table 2. The relations between these quantities and the epsilons, valid in any model of the assumed type, are given in eqs. (3.95a,97a,98,99). For LEP data, we have taken the correlation matrix for $\Gamma_{Z, h}$, $\sigma_{h}$ given by the LEP experiments [33], while we have considered the additional information on $R_{bh}$ and $x$ as independent. We obtain (SLD is also included):

$$
\varepsilon_1 = (4.2 \pm 1.8) \times 10^{-3} - 0.27 \delta \alpha_S
\varepsilon_2 = (-4.9 \pm 4.8) \times 10^{-3} - 0.24 \delta \alpha_S + 0.23 \delta \alpha
$$

(3.103)
\[ \varepsilon_3 = (4.5 \pm 1.8) \times 10^{-3} - 0.17\delta\alpha_S - 0.77\delta\alpha \]
\[ \varepsilon_b = (-0.2 \pm 4.1) \times 10^{-3} - 1.23\delta\alpha_S \]

At this stage, the epsilons have acquired also a dependence on \( \alpha_S(M_Z) \). We have taken \( \alpha_S(M_Z) = 0.118 \) \cite{58} and we have given the variation induced on the epsilons by a corresponding shift of \( \alpha_S(M_Z) \), as defined in Eq. (3.96). The comparison of theory (the SM) and experiment in the planes \( \varepsilon_1 - \varepsilon_3 \) is shown in Fig. 3.6. We see that the inclusion of all LEP quantities does not change the epsilons very much. The effect of a \( \pm 0.007 \) uncertainty on \( \alpha_S(M_Z) \) is included in the quoted error for \( \varepsilon_b \).

---

Figure 3.6: The 1\( \sigma \) ellipse in the plane \( \varepsilon_1 - \varepsilon_3 \) obtained from the data on \( \frac{M_W}{M_Z} \), \( \Gamma_T \), \( \sigma_h, R_h, R_{bh} \) and \( \frac{g_V}{g_A} \) derived from all asymmetries (Table 2), with and without SLD.

To include in our analysis lower energy observables as well, a stronger hypothesis needs to be made: vacuum polarization diagrams are allowed to vary from the SM only in their constant and first derivative terms in a \( q^2 \)-expansion. In such a case, one can, for example, add to the analysis the ratio \( R_v \) of neutral to
charged current processes in deep inelastic neutrino scattering on nuclei [59], the "weak charge" $Q_W$ measured in atomic parity violation experiments on Cs [60] and the measurement of $g_V/g_A$ from $\nu_{\mu}e$ scattering [61] (the final result of CHARM-II corresponds to $s_{2w}^2 = 0.2324 \pm 0.0086$). The expressions of these quantities in terms of the epsilons is given in ref. [51]. In this way one obtains the global fit (also including SLD):

$$\varepsilon_1 = (3.6 \pm 1.7) \times 10^{-3}$$
$$\varepsilon_2 = (-5.3 \pm 4.7) \times 10^{-3}$$
$$\varepsilon_3 = (4.0 \pm 1.7) \times 10^{-3}$$
$$\varepsilon_b = (0.2 \pm 4.0) \times 10^{-3}$$

(3.104)

with the same dependence on $\alpha_s(M_Z)$ and $\alpha(M_Z)$ as in Eqs. (3.103). With the progress of LEP, the low energy data, while important as a check that no deviations from the expected $q^2$ dependence arise, play a lesser role in the global fit. The $\varepsilon_1 - \varepsilon_3$ plot for all data is shown in Fig. 3.7. We observe no drastic change in the epsilons and we take this fact as evidence that no exotic $q^2$ dependence is visible.

Figure 3.7: The 1\(\sigma\) ellipse in the plane $\varepsilon_1 - \varepsilon_3$ obtained from all data.
Note that the present ambiguity on the value of $\alpha(M_Z) = (128.87\pm 0.12) \cdot 1^{-1}$ [34] corresponds to an uncertainty on $\varepsilon_3$ (the other epsilons are not much affected) given by $\Delta \varepsilon_3 = \pm 0.7 \cdot 10^{-3}$. Thus the theoretical error is still comfortably less than the experimental error but the two will become close at the end of the LEP1 phase.

The following final comments can be made.

As is clearly indicated in Fig. 3.7 there is by now a solid evidence for departures from the "improved Born approximation", defined as including the predictions from the tree level SM plus pure QED and pure QCD corrections only, where all the epsilons vanish. Such evidence comes from $\varepsilon_1$ and $\varepsilon_3$, both measured with an absolute error below $2 \cdot 10^{-3}$ and shown to be different from zero at more than the $2\sigma$ level for each of them. In this way one has obtained a strong evidence for pure weak radiative corrections, thus fulfilling one of the explicit goals of the precision electroweak tests. LEP and SLC are now measuring the different components of the radiative corrections.

Of great significance is also the fact that both $\varepsilon_1$ and $\varepsilon_3$ are reproduced in the SM with an appropriate choice of $m_t$ and $m_H$. This can be interpreted as an indirect but nevertheless significant evidence for the description of the electroweak symmetry breaking sector of the theory in terms of fundamental Higgs(es), as in the Standard Model or its supersymmetric extension. This is true in spite of the fact that the dependence of $\varepsilon_1$ and $\varepsilon_3$ on the Higgs mass is rather weak. One should consider in fact that, in most examples of Higgs-less theories that can be found in the literature, $\varepsilon_1$ and $\varepsilon_3$, when they can be computed, show relatively large deviations from the predictions of the SM. In this respect a further reduction of the errors on $\varepsilon_1$ and $\varepsilon_3$, together with an improved direct determination of $m_t$ at the Tevatron, are extremely important. Similarly, it would also be interesting to have a clear evidence for a deviation from zero of the remaining parameters, $\varepsilon_2$ and $\varepsilon_b$. These important goals of the electroweak precision tests are indeed possible in a near future.

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