We show that the nonperturbative quantum transport equations, the ‘Kadanoff-Baym equations’, can be understood as the ensemble average over stochastic equations of Langevin type. For this we couple a free scalar boson quantum field to an environmental heat bath with some given temperature $T$. The inherent presence of noise and dissipation related by the fluctuation-dissipation theorem guarantees that the modes or particles become thermally populated on average in the long-time limit. This interpretation leads to a more intuitive physical picture of the process of thermalization and of the interpretation of the Kadanoff-Baym equations.

1 Motivation

Non-equilibrium many body theory had been traditionally a major topic of research for describing various (quantum) transport phenomena in plasma physics, in condensed matter physics and nuclear physics. Over the last years a lot of interest for non-equilibrium quantum field theory has now emerged also in particle physics. A very powerful diagrammatic tool is given by the ‘Schwinger-Keldysh’ or ‘closed time path’ (CTP) technique by means of non-equilibrium Green’s functions for describing a quantum system also beyond thermal equilibrium. The resulting causal and nonperturbative equations of motion (by various approximations), the so called Kadanoff-Baym (KB) equations, have to be considered as an ensemble average over the initial density matrix characterizing the preparation of the initial state of the system. If the system behaves dissipatively, as a consequence of the famous fluctuation-dissipation theorem, there must exist fluctuations. The Kadanoff-Baym equations have then to be understood as an ensemble average over all the possible fluctuations. This inherent stochastic aspect of the KB equations is what we want to point out and thus provide, as we believe, some new physical insight into its merely complex structure \(^1\). In what follows below we want to point out its intimate connection to Langevin like processes.

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As an elementary reminder of a Langevin process let us first briefly review the description of classical Brownian motion. Consider a heavy ‘Brownian’ particle with mass $M$ placed in a thermal environment obeying an effective Langevin equation, i.e.

$$M \ddot{x} + 2 \int_{-\infty}^{t} dt' \Gamma(t-t') \dot{x}(t') = \xi(t).$$

(1)

Here $\xi(t)$ has to be interpreted as a ‘noisy’ source driving the fluctuations of the Brownian particle. For many applications $\xi(t)$ is completely specified by a Gaussian distribution with zero mean and the correlation kernel $I$:

$$I(t-t') := \langle \langle \xi(t) \xi(t') \rangle \rangle \equiv 2T \Gamma(t-t').$$

(2)

$\langle \langle ... \rangle \rangle$ denotes the average over all possible realizations of the stochastic variable $\xi(t)$. In fact, the simple relation between the dissipation kernel $\Gamma$ and the strength $I$ of the random force $\xi(t)$ just stated is a manifestation of the fluctuation-dissipation theorem and (in the long time limit) is in accordance with the equipartition condition $\langle \langle p^2 \rangle \rangle / (2M) = T/2$.

For a further physical motivation let us return to quantum field theory and already point out some similarities. One of the major present topics in quantum field theory at finite temperature or near thermal equilibrium concerns the evolution and behavior of the long wavelength modes. These modes often lie entirely in the non-perturbative regime. Therefore solutions of the classical field equations in Minkowski space have been widely used in recent years to describe long-distance properties of quantum fields that require a non-perturbative analysis. A justification of the classical treatment of the long-distance dynamics of bosonic quantum fields at high temperature is based on the observation that the average thermal amplitude of low-momentum modes is large. For the low-momentum modes $|p| \ll T$ (and for a weakly coupled quantum field theory) their (Bose) occupation number $n_B$ approaches the classical equipartition limit. The classical field equations should provide a good approximation for the dynamics of such highly occupied modes. However, in a correct semi-classical treatment of the soft modes the hard, i.e. thermal modes cannot simply be neglected, but it should incorporate their influence in a consistent way. In a recent paper it was shown how to construct an effective semi-classical action for describing not only the classical behavior of the long wavelength modes below some appropriate cutoff $k_c$, but taking into account also perturbatively the interaction among the soft and hard modes. By integrating out the ‘influence’ of the hard modes on the two-loop level (for standard $\phi^4$-theory) the emerging semi-classical equations of motion for the soft fields can be derived from
an effective action and become stochastic equations of motion of generalized Langevin type \(^2\), which resemble in their structure the analogous expression to (1). The hard modes act as an environmental heat bath. They also guarantee that the soft modes become, on average, thermally populated with the same temperature as the heat bath. For the semi-classical regime, where \( |\vec{p}| \ll T \), one finds for the ensemble average of the squared amplitude

\[
\frac{1}{V} \langle \langle |\phi(\vec{p})|^2 \rangle \rangle \approx \frac{1}{E_p^2} T \approx \frac{1}{E_p} n_B(E_p) . \tag{3}
\]

Such kind of Langevin description for the non-perturbative evolution of (super-)soft modes (on a scale of \( |\vec{p}| \sim g^2 T \ll T \)) in non Abelian gauge theories has recently been put forward \(^3\). The understanding of the behavior of the soft modes is crucial e.g. for the issue of anomalous baryon number violation due to the diffusion of topological Chern-Simons charge in hot electroweak theory (see eg \(^3\) and references listed therein).

In analogy to the Langevin description stated above we want to sketch in the following (pedagogical) study the effect of the heat bath on the evolution of the system degrees of freedom by means of the ‘closed time path Green’s function’ (CTPGF) technique. For this we discuss a free scalar field theory interacting with a heat bath.

2 Stochastic Interpretation of KB equations

We start with the CTP action for a scalar bosonic field \( \phi \) coupled to an environmental heat bath of temperature \( T \):

\[
S = \int d^4x \frac{1}{2} \left[ \phi^+ (-\Box - m^2) \phi^+ - \phi^- (-\Box - m^2) \phi^- \right] \tag{4}
\]

\[
(-) \int d^4x d^4x' \frac{1}{2} \left[ \phi^+ \Sigma^{++} \phi^+ + \phi^+ \Sigma^{+-} \phi^- + \phi^- \Sigma^{-+} \phi^+ + \phi^- \Sigma^{--} \phi^- \right] .
\]

The system starts to evolve from some initial density matrix. The interaction among the system and the heat bath is stated by an interaction kernel involving a self energy operator \( \Sigma \) resulting effectively from integrating out the heat bath degrees of freedom. Schematically this is sketched in fig. 1. In (4) the self energy contribution from the heat bath is parametrized in the Keldysh notation by the four self energy parts, which can be expressed by means of the standard contributions \( \Sigma^< \) and \( \Sigma^> \). Clearly, this self energy operator is the only quantity which might drive the system towards equilibrium. If the heat bath is at equilibrium, then the Kubo-Martin-Schwinger relation holds:

\[
\Sigma^>(k) = e^{\hbar \omega / T} \Sigma^< (k) . \tag{5}
\]
From (4) it is now straightforward to obtain the equations of motion for
the characteristic two-point functions\(^1\). For the retarded propagator one has

\[(−\Box − m^2 − \Sigma^{\text{ret}})D^{\text{ret}} = δ,\]  \(6\)

where \(\Sigma^{\text{ret}} := \Theta(t_1 − t_2) [\Sigma^> − \Sigma^<]\). Additional dynamical information comes
from the equation of motion of the propagator \(D^<\)

\[−\Box − m^2)D^< − \Sigma^{\text{ret}}D^< − \Sigma^< D^\text{av} = 0.\]  \(7\)

This is just the famous KB equation. (6) and (7) determine the complete and
causal (non-equilibrium) evolution for the two-point functions.

To get more physical insight into the (effective) action (4) and in the
equations of motion we introduce the following real valued quantities:

\[s(x_1, x_2) := \frac{1}{2} \text{sgn}(t_1 − t_2) \left( \Sigma^>(x_1, x_2) − \Sigma^<(x_1, x_2) \right) = s(x_2, x_1),\]  \(8\)

\[a(x_1, x_2) := \frac{1}{2} \left( \Sigma^>(x_1, x_2) − \Sigma^<(x_1, x_2) \right) = −a(x_2, x_1),\]  \(9\)

\[I(x_1, x_2) := −\frac{1}{2i} \left( \Sigma^>(x_1, x_2) + \Sigma^<(x_1, x_2) \right) = I(x_2, x_1).\]  \(10\)

Our notion for \(s\) and \(a\) serves as a reminder for the respective symmetry
properties. It basically represents the standard decomposition of the real and imaginary part of the Fourier transform of the retarded self energy operator \(\Sigma^{\text{ret}}\).

\(s\) yields a (dynamical) mass shift for the \(φ\) modes caused by the interaction
with the modes of the heat bath, while $a$ is responsible for the damping, i.e. 
*disipation* of the $\phi$ fields. The important thing to point out will be that $I$
characterizes the *fluctuations*.

We first note that the CTP action (4) can be written as

$$S = \int d^4x \frac{1}{2} \left[ \phi^+ (-\Box - m^2) \phi^+ - \phi^- (-\Box - m^2) \phi^- \right]$$

$$+ \int d^4x d^4x' \frac{1}{2} \left[ -(\phi^+ - \phi^-)(s + a)(\phi^+ + \phi^-) + i(\phi^+ - \phi^-) I (\phi^+ - \phi^-) \right].$$

This expression is identical to the so called *influence functional* given by
Feynman and Vernon. To the exponential factor $e^{iS}$ in the path integral the $$(s + a)$$
term contributes a phase while the $I$ term causes an exponential damping and thus
signals nonunitary evolution.

The two relevant equations of motion are stated as

$$(-\Box - m^2 - s - a) D^{\text{ret}} = \delta,$$ (12)

$$(-\Box - m^2 - s - a) D^{\text{av}} + (a + iI) D^{\text{av}} = 0.$$ (13)

We see that the last equation is the only one where $I$ occurs.

For the interpretation of $s$, $a$ and $I$ consider the long-time behavior of these
equations. In this case we can assume that the system becomes translational
invariant in time and space and the boundary terms are no longer important.
For the spectral function one immediately finds

$$A(k) := \frac{i}{2} [\bar{D}^{\text{ret}}(k) - \bar{D}^{\text{av}}(k)] = \frac{i \bar{a}(k)}{[k^2 - m^2 - \bar{s}(k)]^2 + |\bar{a}(k)|^2}.$$ (14)

It becomes obvious that $\bar{s} \equiv \Re \Sigma^{\text{ret}}$ contributes an (energy dependent) *mass
shift* while $\bar{a} \equiv \Im \Sigma^{\text{ret}}$ causes the *damping* of propagating modes. $\bar{a}$ is related
to the commonly used damping rate $\bar{\Gamma}$ via

$$\bar{\Gamma}(k) = i \frac{\bar{a}(k)}{k_0}.$$ (15)

For $D^{\text{av}}$ one finds in the long-time limit the relation

$$\bar{D}^{\text{av}}(k) = \bar{D}^{\text{ret}}(k) \Sigma^{\text{av}}(k)$$

$$= \bar{D}^{\text{ret}}(k) [-\bar{a}(k) - i\bar{I}(k)] \bar{D}^{\text{av}}(k) \equiv -2i n(k) A(k),$$ (16)

where, by employing KMS condition (5),

$$n(k) = \frac{\Sigma^{\text{av}}(k)}{\Sigma^{\text{av}}(k) - \Sigma^{\text{ret}}(k)} = \frac{1}{e^{k_0/T} - 1} \equiv n_B(k_0),$$ (17)
which indeed shows that the phase space occupation number in the long-time limit becomes a Bose distribution with the temperature of the heat bath.

It is now very illuminating to explicitly write down the relation between $\bar{a}(k)$ and $\bar{I}(k)$ using the definitions (9) and (10):

$$\bar{I}(k) = \frac{\Sigma^>(k) + \Sigma^<(k)}{\Sigma^>(k) - \Sigma^<(k)} i \bar{a}(k) = \coth\left(\frac{k_0}{2T}\right) i \bar{a}(k).$$

(18)

In the high temperature (classical) limit ($k_0 \ll T$) one gets

$$\bar{I}(k) = \frac{T}{k_0} 2i \bar{a}(k),$$

(19)

or, employing (15),

$$\bar{I}(k) = 2T \bar{\Gamma}(k).$$

(20)

Recalling the discussion of Brownian motion in the introduction this compares favorably well with (2). The physical meaning of $I$ as a ‘noise’ correlator will become obvious. The relation (18) thus represents the generalized fluctuation-dissipation relation from a microscopic point of view by the various definitions of $\bar{I}$, $\bar{a}$ and $\bar{\Gamma}$ through the parts $\Sigma^<$ and $\Sigma^>$ of the self energy.

To see now more closely the connection to stochastic equations we decompose the influence action $S$ as given in (11) in its real and imaginary part and write for the corresponding generating functional

$$Z[j^+, j^-] := \int D[\phi^+, \phi^-] \rho[\phi^+, \phi^-] e^{iS[\phi^+|\phi^-] + ij^+ \phi^+ + ij^- \phi^-}$$

$$= \int D[\phi^+, \phi^-] \rho[\phi^+, \phi^-] e^{iRS[\phi^+|\phi^-] + ij^+ \phi^+ + ij^- \phi^- - \frac{1}{2}(\phi^+ - \phi^-) I (\phi^+ - \phi^-)}$$

$$= \frac{1}{\bar{N}} \int D[\xi] e^{-\frac{1}{2}I^{-1} \xi} \int D[\phi^+, \phi^-] \rho[\phi^+, \phi^-] e^{iRS[\phi^+|\phi^-] + ij^+ \phi^+ + ij^- \phi^- + i \xi (\phi^+ - \phi^-)}$$

$$= \frac{1}{\bar{N}} \int D[\xi] e^{-\frac{1}{2}I^{-1} \xi} Z'[j^+ + \xi, j^- - \xi] \equiv \langle\langle Z'[j^+ + \xi, j^- - \xi]\rangle\rangle$$

(21)

with $\bar{N} := \int D[\xi] e^{-\frac{1}{2}I^{-1} \xi}$. The action entering the definition of $Z'$ is no longer $S$, but only the real part of the influence action (11). The generating functional $Z'[j^+, j^-]$ can thus be interpreted as a new stochastic generating functional $Z'[j^+ + \xi, j^- - \xi]$ averaged over a random Gaussian (noise) field $\xi$ with the width function $I$, i.e.

$$\langle\langle O\rangle\rangle := \frac{1}{\bar{N}} \int D[\xi] O e^{-\frac{1}{2}I^{-1} \xi}.$$  

(22)
From the last definition we find that the (ensemble) average over the noise field vanishes, i.e. $\langle \xi \rangle = 0$, while the noise correlator is given by

$$\langle \langle \xi \rangle \rangle = I. \quad (23)$$

From the stochastic functional $Z'$ a Langevin equation for a classical $\phi$ field can now readily be derived. Noting that the fields $\langle \phi^+ \rangle_\xi$ on the upper branch and $\langle \phi^- \rangle_\xi$ on the lower branch are equal (and denoted as $\phi_\xi$ in the following), its equation of motion derived from $Z'$ takes the form

$$(-\Box - m^2 - s) \phi_\xi - a \phi_\xi = -\xi. \quad (24)$$

This, indeed, represents a standard Langevin equation. The spatial Fourier transform of the Langevin equation (24) then takes the form

$$\ddot{\phi}_\xi(\vec{k}, t) + (m^2 + \vec{k}^2 - 2\Gamma(\vec{k}, \Delta t = 0))\phi_\xi + 2 \int_{-\infty}^{t} dt' \Gamma(\vec{k}, t - t') \dot{\phi}_\xi(\vec{k}, t') = \xi(\vec{k}, t). \quad (25)$$

The analogy between this Langevin equation (25) and the one for a single classical oscillator is obvious. The important difference, however, is the fact that the corresponding relations (18) and (2) between the respective noise kernel $I$ and friction kernel $\Gamma$ only agree in the high temperature limit.

One can further ask to what extend the classical equations of motion (24) together with (23) are an approximation for the full quantum problem given by the equation of motion (13) for $D^\leq$. Inverting (24) one finds for the correlation function in the long-time limit

$$-i\langle \langle \phi^+ \rangle_\xi \langle \phi^- \rangle_\xi \rangle = -i D^{\text{ret}} \langle \langle \xi \rangle \rangle D^{\text{av}} = -i D^{\text{ret}} I D^{\text{av}}. \quad (26)$$

Note that (26) is indeed the relation (3) advocated in the introduction to hold in the (semi-)classical regime. This one has to compare with the full quantum correlation function $D^\leq$ of (16). One thus has that $(-\bar{a} - i\bar{I})$ is approximated by $-i\bar{I}$. Of course this is justified, if $|\bar{a}| \ll \bar{I}$ holds. Using the microscopic quantum version (18) of the fluctuation-dissipation theorem this is equivalent to $\coth (k_0^2 T) \gg 1$. Thus in the high temperature limit or – turning the argument around – for low frequency modes, i.e. for $k_0 \ll T$, the classical solution yields a good approximation to the full quantum case. To be more precise: In simulations one has to solve the classical Langevin equation (24) and calculate $n$-point functions by averaging over the random sources.

One can also write down the equations of motion for the quantum two-point functions with external noise by introducing 'noisy' two-point propagators. Averaging the equation of motion over the noise fields according to (22)
one indeed rederives the KB equation. This demonstrates that the KB equation can be interpreted as an ensemble average over fluctuating fields which are subject to noise, the latter being correlated by the sum of self energies $\Sigma^<$ and $\Sigma^>$, i.e. from a transport theoretical point of view the sum of production and annihilation rate. We want to note once more that the ‘noisy’ or fluctuating part denoted by $I$ inherent to the structure of the KB equation (7) guarantees that the modes or particles become correctly (thermally) populated, as can be realized by inspecting (16).

We close our discussion by noting that one can also pursue to derive a standard kinetic transport equation for the (semi-classical) phase-space distribution $f(\vec{x}, \vec{k}, t)$ including fluctuations $^1$. The derived kinetic transport process has the structure of the phenomenologically inspired Boltzmann-Langevin equation. Our approach carried out in $^1$ has to be considered as a clear derivation from first principles. Indeed it shows (nearly) a one to one correspondence to the phenomenologically introduced scheme. However, also some severe interpretational difficulties in the interpretation of the fluctuating phase-space density remain. We refer the interested reader to our discussion in $^1$.

3 Some further conclusions

In our discussions we have elucidated on the stochastic aspects inherent to the (non-) equilibrium quantum transport equations. We have isolated a term denoted by $I$ which solely characterizes the (thermal and quantum) fluctuations inherent to the underlying transport process. By introducing a stochastic generating functional the emerging stochastic equations of motion can then be seen as generalized (quantum) Langevin processes. What is changed, if we replace our toy model of a free system coupled to an external heat bath by a self-coupled and thus nonlinear closed system? In an interacting field theory of a closed system the KB equations formally have exactly the same structure as in our toy model. The important difference, however, is that the self energy operator is now described fully (within the appropriate approximative scheme) by the system variables, i.e. it is expressed as a convolution of various two-point functions. Hence, an underlying simple stochastic process, as in our case an external stochastic Gaussian process, cannot really be extracted. However, we emphasize that the emerging structure of the KB equations is identical. The decomposition of the self energy operator into its three physical parts (mass shift $s$, damping $a$, and fluctuation kernel $I$) can immediately be taken over. Hence these three parts keep their clear physical meaning also for a nonlinear closed system.

One can also nicely demonstrate how so called pinch singularities $^4$ are
regulated within the non-perturbative context of the thermalization process. These singularities do (and have to) appear in the perturbative evaluation of higher order diagrams within the CTP description of non-equilibrium quantum field theory. They are simply connected to the standard divergence in elementary scattering theory. The occurrence of pinch singularities signals the occurrence of (onshell) damping or dissipation. This necessitates in the description of the evolution of the system by means of non-perturbative transport equations.

As a further application (discussed in the talk) for the direct use of semiclassical Langevin equations we want to mention the recent work in 5: By applying a microscopically motivated Langevin description of the linear sigma model, one can investigate the stochastic evolution of a so called disoriented chiral condensate (DCC) in a rapidly expanding system, expected to occur in ultrarelativistic heavy ion collisions. Within such an approach one finds that an experimentally feasible DCC, if it does exist in nature, has to be a rare event, but still occurring with some finite and nonvanishing probability. The statistical distribution of final emitted pion number out of domains shows a striking nonpooissonian and nontrivial behaviour. One should indeed interpret those particular rarely occurring events as semi-classical ‘pion bursts’ similar to the mystique Centauro candidates. A further analysis of this unusual distribution by means of the cumulant expansion shows that the reduced higher order factorial cumulants exhibit an abnormal, exponentially increasing tendency and thus serves as a new and powerful signature. The occurrence of a rapid chiral phase transition (and thus DCCs) might then probably only be identified experimentally by inspecting higher order factorial cumulants $\theta_m \ (m \geq 3)$ for taken distributions of low momentum pions.

References