Black hole constraints on the running-mass inflation model

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The running-mass inflation model, which has strong motivation from particle physics, predicts density perturbations whose spectral index is strongly scale-dependent. For a large part of parameter space the spectrum rises sharply on short scales. In this paper we compute the production of primordial black holes, using both analytic and numerical calculation of the density perturbation spectra. Observational constraints from black hole production are shown to exclude a large region of otherwise permissible parameter space.

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I. INTRODUCTION

Particle physics models of inflation based on supergravity theories are plagued by the so-called $\eta$-problem [1], which states that even if at tree-level one has a potential flat enough to support inflation, this flatness is typically spoilt by quantum corrections, which make the mass of the putative inflaton field of order $H^2$ ($H$ being the Hubble parameter) and ruin slow-roll inflation. An elegant proposal to circumvent this is the so-called running-mass model of inflation, introduced by Stewart [2,3], where the flatness of the potential arises because of the quantum corrections, which serve to flatten the potential over a significant region where inflation can then take place.

Because the flatness is brought about by a cancellation of the intrinsic curvature of the potential against the quantum corrections, only a limited portion of the potential can support slow-roll inflation, as is necessary to generate the approximately flat power spectrum seen by the COBE satellite. Well away from COBE scales, one typically expects to see dramatic deviations from near scale-invariance as the slow-roll regime breaks down. Copeland et al. [4] examined the possibility that this breakdown of scale-invariance might be detectable through scale-dependence of the spectral index of primordial perturbations, and more recently the model has been extensively explored by Covi, Lyth and collaborators [5–7] in a series of papers investigating its viability both from a theoretical standpoint and in confrontation against large-scale structure data.

In this paper we examine constraints arising from the more radical departures from scale-invariance which may take place towards the end of inflation. In much of parameter space, the spectrum rises sharply on short scales, which can give rise to production of primordial black holes (PBHs). These are strongly constrained by observation and, as we will see, a significant region of otherwise viable parameter space is excluded.

II. THE RUNNING-MASS MODEL

Whether or not a potential $V(\phi)$ can support slow-roll inflation can be judged via the slow-roll parameters [8]

\[
\epsilon_\nu \equiv \frac{1}{2} M_P^2 \left(\frac{V'}{V}\right)^2; \quad \eta_\nu \equiv M_P^2 \frac{V''}{V},
\]

(1)

where primes denote differentiation with respect to $\phi$ and whose values should be much less than unity. Under such circumstances slow-roll inflation can proceed, and gives rise to density perturbations with an approximately scale-invariant spectrum.

Within the context of softly-broken global supersymmetry, the false vacuum dominated potential

\[
V = V_0 \left[1 + \frac{1}{2} \mu^2 \frac{\phi^2}{M_P^2} + \ldots\right],
\]

(2)

arises naturally [1]. However it will not in general lead to slow-roll inflation, because supergravity corrections lead to $|\mu^2| = |\eta_\nu| \approx 1$ in Planck units. In the scenario proposed by Stewart [2,3], the inflaton has gauge couplings to vector or chiral superfields, and one-loop quantum corrections flatten the potential, corresponding to a running of the effective mass with the scalar field value

\[
\mu^2 \equiv \mu^2 \left[\mu_0^2, A_0, \alpha_0 \ln \frac{\phi}{M_P}\right],
\]

(3)

where $\mu_0^2$ represents the inflaton mass squared, $A_0$ is the mass squared of the gaugino appearing in the loop, both evaluated at the Planck scale, and $\alpha_0$ is the gauge coupling. The functional form of $\mu^2(\phi)$ is obtained by solving the relevant renormalization group equations. For definiteness, we consider the inflaton potential [2,3,5,6]

\[
\frac{V}{V_0} = 1 - \frac{\phi^2}{2 M_P^2} \left\{\mu_0^2 + A_0 \left[1 - \frac{1}{\left(1 + \alpha_0 \ln \frac{\phi}{M_P}\right)^2}\right]\right\}.
\]

(4)
The Planck scale, the desired cancellation occurs between the potential reverts to the form of Eq. (2). Far below the potential the model can be found in Refs. [5,6].

An attractive feature of this model is that its parameters take on natural values; for a positive gauge coupling we have $A_0$ and $\mu_0^2$ both positive and of order unity, while for a negative gauge coupling we have $A_0$ negative and the model is subject to the constraint $|A_0| > \mu_0^2 + 1$, which is applied to ensure that the inflaton mass changes sign before reaching the end of inflation. Full details of which is applied to ensure that the inflaton mass changes sign before reaching the end of inflation. Full details of the model can be found in Refs. [5,6].

The inflaton field starts off near the maximum of the potential* and rolls towards the origin, corresponding to model (i) of Ref. [6] and as shown in Fig. 1. In the case of $\alpha_0 > 0$, the potential has an unphysical pole at $\ln(\phi/M_P) = 1/\alpha_0$ and should not be trusted in this region of strong coupling. It is not necessary, though, to evolve $\phi$ to such small field values.

Throughout this calculation we use the Hubble-slow-roll parameters, defined as [10]

\[
\epsilon \equiv 2M_P^2 \left( \frac{H'}{H} \right)^2; \quad \eta \equiv 2M_P^2 \frac{H''}{H}; \\
\xi \equiv 2M_P^2 \left( \frac{H'H''}{H^2} \right)^{1/2},
\]

where the fundamental quantity is now taken to be the Hubble parameter $H$ and its derivatives, rather than the potential.

The value of $\eta$ starts off negative near the maximum of the potential and runs through zero until the end of slow-roll inflation is reached, which we define to be

\[
\eta(\phi_{\text{end}}) = 1.
\]

At this point the inflaton potential is still dominated by the $V_0$ term, so it is assumed that the field must decay via some hybrid inflation mechanism when $\phi$ falls below a critical value $\phi_c$, in order that reheating occurs to restore the standard cosmology.

### III. Computing the Perturbation Spectra

Our main focus is the perturbations near the end of inflation, where the slow-roll approximation will be poor. It is therefore imperative that the accuracy of calculations is checked numerically. There are two aspects to this; numerical calculation of the classical scalar field dynamics, and numerical calculation of the perturbation equations.

The numerical calculation of the classical evolution is important in determining which part of the potential generates the perturbations seen on cosmological scales. When the slow-roll approximation begins to break down, commonly-used expressions such as that for the number of $e$-foldings $N$ can lose their accuracy and the numerical evolution can lead to some corrections to the analytic results. In general, $N$ and the wavemode $k$ leaving the horizon at that epoch are related by

\[
N(k) = N_{\text{COBE}} - \ln \frac{k}{k_{\text{COBE}}},
\]

where $N_{\text{COBE}}$ is defined throughout as the number of $e$-foldings before the end of inflation when our present Hubble radius, in comoving units, equalled the Hubble radius during inflation. This expression neglects the variation of $H$, which is valid as long as $\epsilon_v$, is small even when $\eta_v$ is not.

To numerically compute the perturbations, we use the Mukhanov formalism [11,12] as described by Grivell and Liddle [13], to which we refer the reader for details. The approach involves a numerical solution both of the classical homogeneous equations of motion and of the equations describing linear perturbations, with the full power spectrum being built up mode-by-mode. In order to evaluate the power spectrum, one has to follow the modes until they are well outside the horizon, where their amplitude becomes constant. This becomes problematic once one reaches very close to the end of inflation, where this asymptotic regime is not reached. In fact, in the case of the running-mass model it proves difficult to numerically evolve the mode evolution much more than around one $e$-folding beyond the $\eta = 1$ point, since the slow-roll parameters $\eta$ and $\xi^2$ are growing so rapidly at this point.

*Note that initial conditions near the maximum of the potential are well motivated by the ‘topological inflation’ idea [9]; the initial conditions inevitably have the field on different sides of the maximum in different regions of space, and hence crossing the maximum in the interpolating regions.
However, as long as one uses the numerical solution for the classical background evolution, it turns out we do not need to compute the perturbations numerically; the extended slow-roll approximation of Stewart and Lyth [14] proves perfectly adequate. This gives the perturbation amplitude as

$$\delta_H(k) \simeq [1 - (2C + 1)\epsilon + C\eta] \frac{H^2}{16\pi M_P^2} \left| \frac{d}{dt} \right|_{k=aH}, \quad (8)$$

where $C = -2 + \ln 2 + b \simeq -0.73$ and $b$ is the Euler–Mascheroni constant, and gives sufficiently accurate results even when $\eta$ becomes large. Fig. 2 shows a comparison of numerical simulation with the slow-roll and Stewart–Lyth predictions.

The error in the Stewart–Lyth expression is expected to be $O(\xi^2)$. It is known to underestimate the perturbation amplitude when $\eta \simeq 1$ and $\epsilon \ll 1$ [13], which makes our PBH constraints conservative. A further consequence of the smallness of $\epsilon$,

$$\epsilon = \frac{1}{2} \frac{\eta^2}{M_P^2}, \quad (9)$$

is that the gravitational waves from this model are strongly suppressed [8].

We use the COBE normalization scheme of Ref. [15], setting $\Omega_0 = 0.35$ and $\Omega_\Lambda = 0.65$. One subtlety that must be remembered is to divide by the growth suppression factor in order to convert the present power spectrum normalization to the primordial one. This choice of cosmological parameters has the effect of raising the primordial value of $\delta_H(k_{COBE})$ by a factor of around 2.5 compared to a critical-density model. In this scheme, normalization occurs at the scale $k = 70_0 H_0 = 7 h/3000 \text{ Mpc}^{-1}$.

A typical power spectrum is shown in Fig. 2, taken from a region of parameter space that we will show to be excluded by PBH constraints. The scale-dependence of the spectral index is clearly seen. By virtue of the $N(k)$ relation of Eq. (7), a change in $N_{COBE}$ corresponds simply to a translation of the power spectrum to a new end point given by $k_{end} = e^{N_{COBE}} h/3000 \text{ Mpc}^{-1}$ (though it must be renormalized to COBE). A change in the condition for the end of slow-roll inflation also results in a translation and renormalization of the power spectrum.

As we shall see in Section IV, the condition for the end of slow-roll inflation, Eq. (6), is one of two parameters that affect the severity of the PBH constraints the most, the other being $N_{COBE}$.

In Section IV we consider the cases $N_{COBE} = 45$, which corresponds to instant reheating after the end of slow-roll inflation, and $N_{COBE} = 25$, which may result if the end of slow-roll is followed by a burst of fast-roll and/or thermal inflation.

### IV. CONSTRAINTS FROM PRIMORDIAL BLACK HOLE PRODUCTION

#### A. The black hole constraint

The astrophysical details of PBH constraints have been studied in detail elsewhere [16]. Over a wide range of mass scales, the observational constraint on the black hole formation rate is that no more than around $10^{-20}$ of the mass of the Universe can be channeled into black holes. For our purposes it is sufficient to ignore the details of the constraints, and simply adopt this level, as the black hole production rate is enormously sensitive to the amplitude of perturbations.

In computing the black hole formation rate, the quantity which is of interest is the matter dispersion $\sigma$, which is defined, in the usual way (see e.g. Ref. [8]), as the matter distribution smoothed over some length scale $R$,

$$\sigma^2(R,t) = \int_0^\infty \left( \frac{k}{aH} \right)^4 \delta_H^2(k) W^2(kR) \frac{dk}{k}. \quad (10)$$

We will take $W(kR)$ to be a gaussian filter. The factor $10/9$ appears because we are interested in perturbations in the radiation era rather than the usual matter era.

At the end of slow-roll inflation we have $\eta = 1$, and so the power spectrum is rising steeply with a spectral index $n - 1 \simeq 2\eta \simeq 2$. Approximating $\delta_H^2(k)$ as a power law at the end of slow-roll inflation,

$$\delta_H^2(k) \simeq \delta_H^2(k_{end})(k/aH)^{n-1}, \quad (11)$$

and setting $aH = 1/R$, we can evaluate the dispersion Eq. (10) at horizon crossing,

$$\sigma^2_{hor}(k_{end}R) \simeq \left( \frac{10}{9} \right)^2 \delta_H^2(k_{end})(k_{end}R)^4 I(n), \quad (12)$$

FIG. 2. An example power spectrum, taking $\alpha_0 = 0.01$, $\Lambda_0 = 1.0$, $\nu_5 = 0.5$ and $N_{COBE} = 45$. The numerical calculation of the perturbation amplitude breaks down towards the end of inflation. The end point of the power spectrum ($k_{end} = e^{N_{COBE}} h/3000 \text{ Mpc}^{-1}$) is defined to be where $\eta = 1$. The position where $\eta = 0.5$ is shown for comparison.
where $I(n)$ is a numerical factor of order unity which depends on the spectral index. The length scale at the end of slow-roll inflation also provides the natural scale over which to smooth the power spectrum since it is the scale on which black holes will predominantly form. Setting $k_{\text{end}} R = 1$ we can evaluate $I$ to be

$$I = \int_{k_{\text{end}}}^{1} k^{(n+2)} W^2(k k_{\text{end}} R) dk = \frac{1}{2} \gamma \left((n+3)/2,1\right), \quad (13)$$

where $\gamma[\alpha, x]$ is the incomplete gamma function. For example, with $n - 1 = 3$ we have $I \approx 0.067$, $n - 1 = 2$ we have $I \approx 0.080$, while for $n - 1 = 1$ we have $I \approx 0.100$. We note that in the limit of large $n$, holding $\delta_H(k_{\text{end}})$ constant, the contribution of this spike to $\sigma_{\text{hor}}$ is suppressed by the numerical factor $I$. We can see immediately that under the power-law approximation of Eq. (11), the dispersion $\sigma_{\text{hor}}$ only depends weakly on the exact value of the spectral index at the end of slow-roll inflation which we take to be a nominal and conservative $n = 3$.

The main dependence of $\sigma_{\text{hor}}$ is on the perturbation amplitude $\delta_H(t)$, although in our case, and for any sharply rising power spectrum, $\delta_H(t)$ depends strongly on the exact condition for the point where slow-roll inflation ends.

As shown in Ref. [17], the black hole constraint across all scales simply amounts to

$$\sigma_{\text{hor}} \lesssim 0.04. \quad (14)$$

This is sufficient to ensure that no more than $10^{-20}$ of the mass density of the Universe is channelled into black holes. The constraint on $\sigma_{\text{hor}}$ is expected to be accurate to within a factor of 2, which is small compared to the 2–3 orders of magnitude that the power spectrum can rise between the $\eta = 1/2$ and $\eta = 1$ points. This uncertainty is therefore much less important than the uncertainty of the end-point of inflation, which we discuss further below.

As well as the observational constraints on the model, there is a self-consistency constraint which must be satisfied, which is to ensure that the inflationary energy scale, once normalized to COBE, is high enough to permit the claimed number of $e$-foldings $N_{\text{COBE}}$. If we conservatively assume instant reheating after inflation, and that the radiation era is not punctuated by episodes of thermal inflation or temporary matter domination, the upper bound on the number of $e$-foldings that can take place is

$$N_{\text{COBE}} < 48 + \ln(V_0^{-1/4}/10^{10}\text{GeV}), \quad (15)$$

which requires $V_0/M_P^4 \gtrsim 10^{-36}, 10^{-72}$ for $N_{\text{COBE}} = 45, 25$ respectively.

**B. Results**

The results for $N_{\text{COBE}} = 45, 25$ and a positive gauge coupling, $\alpha_0 = 0.01$, are shown in Fig. 3. We plot three different values of $\sigma_{\text{hor}}$; the central one is the best guess

![FIG. 3. Parameter space constraints for $\alpha_0 = 0.01$ for two choices of $N_{\text{COBE}}$. The solid lines show the spectral index on COBE scales. The dashed lines are $\sigma_{\text{hor}} = 0.02, 0.04, 0.08$ contours, where $\sigma_{\text{hor}}$ has been evaluated at the end of slow-roll inflation. Parameter space below this region is excluded, and these models will violate the bound on $\sigma_{\text{hor}}$ before the end of slow-roll inflation. From Eq. (15), instant reheating requires $V_0/M_P^4 \gtrsim 10^{-36}$ for $N_{\text{COBE}} = 45$, indicated by the dot-dashed contour; the parameter space above this contour is excluded.](image-url)
FIG. 4. As Fig.3, but for \( \alpha_0 = -0.01 \). The thick line to the right is a bound on the allowed values of the parameters \( |A_0| > \mu_0^2 + 1 \). The dashed lines, reading from left to right, are the \( \sigma_{\text{hor}} = 0.02, 0.04, 0.08 \) contours, and the region to the right of these contours is excluded by PBH constraints. From Eq. (15), instant reheating requires \( V_0/M_p^4 \gtrsim 10^{-36} \) for \( N_{\text{COBE}} = 45 \) indicated by the dot-dashed contour; the parameter space above this contour is excluded.

at where the constraint lies and the others indicate the uncertainty. We see that the PBH constraint actually quite closely follows lines of constant \( n \) on COBE scales, enabling us to use this to summarize the constraint. In the case of \( N_{\text{COBE}} = 45 \) we find that a large amount of otherwise viable parameter space is excluded: models with \( n \gtrsim 1.1 \) are ruled out. This should be compared with PBH constraints on inflation models with constant spectral index, for which the end result is \( n \gtrsim 1.25 \) are excluded [16]. It is of course not surprising that the constraint on \( n \) should be stronger for the running-mass model whose spectral index increases as a function of wavenumber \( k \).

For \( N_{\text{COBE}} = 25 \) the PBH constraints are less severe, models with \( n \gtrsim 1.3 \) being excluded. The simplest explanation is that the mass runs for fewer \( \epsilon \)-foldings leading to a safer period of slow-roll inflation. In fact it is a combination of two factors that make the \( N_{\text{COBE}} = 25 \) case safer. Firstly, reducing \( N_{\text{COBE}} \) has the effect of translating the spectral index contours away from the region of parameter space previously excluded: for a given spectral index contour, the values of \( A_0 \) and \( \mu_0^2 \) must increase as \( N_{\text{COBE}} \) decreases to ensure that the running of the mass up to \( \eta = 1 \) is faster. Secondly, when we renormalize the new spectra (for given values of \( \mu_0^2 \) and \( A_0 \)) to COBE, the overall amplitude at the end of slow-roll inflation will be reduced for all models with \( n > 1.0 \) as compared to the \( N_{\text{COBE}} = 45 \) case, because the spectral index \( n \) is an increasing function of \( N \). Therefore, since the \( \sigma_{\text{hor}} = 0.04 \) contour lies in the region \( n > 1.0 \) for \( N_{\text{COBE}} = 25 \), the excluded region of parameter space shrinks for the \( N_{\text{COBE}} = 25 \) case.

Next we look at the results for a negative gauge coupling, \( \alpha_0 = -0.01 \), which are shown in Fig. 4. For \( N_{\text{COBE}} = 45 \), the PBH constraints are more severe, ruling out models with \( n \gtrsim 1.0 \). This result is related to the running strength throughout inflation, given by [18] (neglecting terms in \( \epsilon \))

\[
\frac{dn}{d\ln k} \simeq -2\xi^2,
\]  \( (16) \)

Over observable cosmological scales the running strength given by \( -\xi^2 \) is generally greater in the negative gauge coupling case (for a given spectral index contour \( n \)) resulting in a larger value of \( \eta \) throughout slow-roll inflation. For instance, for \( n - 1 = 0 \) we have, over observable scales [5],

\[
\frac{dn}{d\ln k} \simeq 8A_0^2\alpha_0^2 \left( 1 + \frac{\mu_0^2}{A_0} \right)^3.
\]  \( (17) \)

Towards the end of slow-roll inflation the running becomes stronger in the positive gauge coupling case as the field approaches our fixed end point, \( \eta = 1 \). For \( N_{\text{COBE}} = 25 \) the spectral index contours are once again shifted close to the point where the PBH constraints cease to be very interesting, with \( n \gtrsim 1.3 \) being excluded.

Finally we would like to know what effect varying the value of \( \alpha_0 \) has on the PBH constraints. Given that the \( \sigma_{\text{hor}} \) contours are approximately parallel to the spectral index contours, \( n_{\text{hor}} \), we can reduce the dimensionality of this calculation, and simply assign to each \( \alpha_0 \) some critical value of the spectral index (on COBE scales), \( n_{\text{crit}} \), above which the model is excluded. We will assume the constraint is \( \sigma_{\text{hor}} < 0.04 \). The results are shown in Fig. 5 and illustrate the trends of the PBH constraints when we vary \( \alpha_0 \). For \( \alpha_0 > 0 \) the PBH constraints become less restrictive as \( \alpha_0 \) is increased, since the other parameters of the model take on smaller values, which has the effect of reducing the overall running strength. For \( \alpha_0 < 0 \) the PBH constraints become more restrictive as \( |\alpha_0| \) is increased, since the other model parameters remain fairly static, and the overall running strength becomes greater.
using the relation [6]

is related to the coupling strengths involved and given values of for a numerical calculation of the PBH constraints. For the linear approximation breaks down, hence the need cosmological scales, but not away from these scales where are still enough to describe the density perturbation over inflation. In more general models these two quantities integration constant related to the endpoint of slow-roll constraints on the 

Before ending, we need to comment on our choice for the end of slow-roll inflation given by Eq. (6); as we have remarked the constraints can be highly sensitive to this and we need to ensure we are being conservative. There is no kinematical reason why inflation cannot proceed when \( \epsilon < 1 \) and \( \eta \lesssim 1 \), although we know that this situation can be tolerated for no more than a few e-folds, given a COBE-normalized spectrum of perturbations. This is just restatement of the \( \eta \)-problem, and indeed is observed in our simulations where inflation always proceeds to the \( \eta = 1 \) point and beyond. However, as we move into the regime where \( \eta \gtrsim 1 \) we find that the running strength given by \( -\epsilon^2 \) begins to blow up, marking the failure of the one-loop approximation. This suggests it is dangerous to try and proceed further along the potential even though the numerical simulations show inflation continuing and the spectrum continuing to rise. Thus, evolving the inflaton to \( \phi_{\text{end}} \) where \( \eta = 1 \) but not beyond ap-

\[
n - 1 = 2\sigma e^{-CN_{\text{COBE}}} - 2c. \quad (18)
\]

We have investigated primordial black hole constraints within the context of the failure of slow-roll inflation, focusing on the well-motivated running-mass model of inflation which features a strong scale-dependence of the spectral index [7]. Although applying to the amplitude of perturbations on very short scales, to a good approximation the constraint can be represented as a constraint on \( n_{\text{COBE}} \), the spectral index on the largest observable scales. The constraint depends strongly on the number of e-foldings \( N_{\text{COBE}} \) between the production of those perturbations and the end of inflation, and, as with models with a constant spectral index, the constraint becomes weaker as \( N_{\text{COBE}} \) is reduced.

We have shown that a significant region of the parameter space of the model, viable under other constraints, is excluded by excess production of black holes. This demonstrates the importance of evaluating the density perturbation spectrum not just across astrophysical scales but also right to the end of inflation. In models where the slow-roll approximation holds accurately only over a limited range of scales, such as the running-mass model, there will be strong deviations from scale-invariance towards short scales. In models where the deviation takes the form of a strongly blue spectrum, excessive black hole production is always likely to be a danger.

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