BLACK AND SUPER P-BRANES IN DIVERSE DIMENSIONS

M. J. Duff†

Center for Theoretical Physics, Physics Department, Texas A & M University
College Station, TX 77843

J. X. Lu‡

CERN, Theory Division, CH-1211, Geneva 23, Switzerland

Abstract

We present a generic Lagrangian, in arbitrary spacetime dimension $D$, describing the interaction of a dilaton, a graviton and an antisymmetric tensor of arbitrary rank $d$. For each $D$ and $d$, we find “solitonic” black $p$-brane solutions where $p = d - 1$ and $d = D - d - 2$. These solutions display a spacetime singularity surrounded by an event horizon, and are characterized by a mass per unit $p$-volume, $\mathcal{M}_d$, and topological “magnetic” charge $g_d$, obeying $\kappa \mathcal{M}_d \geq g_d / \sqrt{2}$. In the extreme limit $\kappa \mathcal{M}_d = g_d / \sqrt{2}$, the singularity and event horizon coalesce. For specific values of $D$ and $d$, these extreme solutions also exhibit supersymmetry and may be identified with previously classified heterotic, Type IIA and Type IIB super $p$-branes. The theory also admits elementary $p$-brane solutions with “electric” Noether charge $\epsilon_d$, obeying the Dirac quantization rule $\epsilon_d g_d = 2\pi n$, $n =$ integer. We also present the Lagrangian describing the theory dual to the original theory, whose antisymmetric tensor has rank $\tilde{d}$ and for which the roles of topological and elementary solutions are interchanged. The super $p$-branes and their duals are mutually non-singular. As special cases of our general solution we recover the black $p$-branes of Horowitz and Strominger ($D = 10$), Guven ($D = 11$) and Gibbons et al ($D = 4$), the $N = 1$, $N = 2a$ and $N = 2b$ super-$p$-branes of Dabholkar et al ($4 \leq D \leq 10$), Duff and Stelle ($D = 11$), Duff and Lu ($D = 10$) and Callan, Harvey and Strominger ($D = 10$), and the axionic instanton of Rey ($D = 4$). In particular, the electric/magnetic duality of Gibbons and Perry in $D = 4$ is seen to be a consequence of particle/sixbrane duality in $D = 10$. Among the new solutions is a self-dual superstring in $D = 6$.

CERN-TH.6675/93
June 1993

† Work supported in part by NSF grant PHY-9106593
‡ Supported in part by a World Laboratory Fellowship
1. Introduction

Supersymmetric extended objects [1] are interesting for a variety of reasons. First, they correspond to the extreme mass = charge limit [2] of the black $p$-branes [3,4,5,6,7], which are higher dimensional analogues of black holes. These super $p$-branes, stable by virtue of the supersymmetry which emerges in this limit, might thus describe the end point of Hawking radiation. Secondly, they emerge as topological defects [8,9] of supersymmetric field theories, and might thus have interesting cosmological consequences. In particular they provide soliton solutions of $N = 1$ supergravity-Yang-Mills, $N = 2$ supergravity and $N = 2B$ supergravity in $D = 10$ [10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25] which are the field theory limits of the $D = 10$ heterotic, Type IIA and Type IIB superstrings, respectively. In some cases, one can show that they correspond, in fact, to exact conformal field theories [18,19,20,4] and must therefore be taken just as seriously by string theorists as magnetic monopoles are by grand unified theorists: one cannot buy superstrings without buying super $p$-branes in the same package! The final, more speculative, but perhaps most intriguing reason, is the possibility that they may provide a dual description of superstrings. For example, there is a mounting body of evidence to suggest that in $D = 10$, the heterotic superstring is dual to the heterotic fivebrane [26,11] with the strongly coupled string corresponding to the weakly coupled fivebrane [11,13]. The study of super $p$-branes might thus throw light on the strong coupling regime of string theory.

The plan of the paper is as follows: In section 2, we write down a general action in $D$ spacetime dimensions describing the interaction of an antisymmetric tensor potential of rank $d$ with gravity and a dilaton. We allow these fields to couple to an elementary $d$-dimensional extended object, (a “$p$-brane”, with $d = p + 1$) and define an “electric” Noether charge associated with it. In section 3, we show how the combined field equations admit solutions describing such elementary objects, in much the same way as Dabholkar et al [10] showed how an elementary string emerges as a solution of supergravity coupled to a string $\sigma$-model source. As described in section 4, one may establish a Bogolmol’nyi bound between mass per unit $p$-volume $M_d$ of the $p$-brane and the Noether charge $\epsilon_d$, and demonstrate that these elementary solutions saturate the bound, and are thus seen to be classically stable. One may also demonstrate a “no-static-force” condition by showing that
the mutual gravitational-dilaton attraction of two such $p$-branes of the same orientation is exactly cancelled by an equal and opposite contribution from the antisymmetric tensor. This permits the construction of stable multi-$p$-brane solutions.

In addition to the singular elementary $(d-1)$-brane solutions carrying non-zero “electric” Noether charge $e_d$, the theory also admits non-singular soliton $(\tilde{d}-1)$-brane solutions, where $\tilde{d} = D - d - 2$. As described in section 5, these solutions are dual to the elementary solutions and carry a non-zero “magnetic” topological charge $g_{\tilde{d}}$, obeying the Dirac quantization rule [27,28]

\[ e_d g_{\tilde{d}} = 2\pi n, \quad n = \text{integer}. \quad (1.1) \]

In section 6 we consider the theory dual to the theory of section 2, for which the roles of antisymmetric tensor field equations and Bianchi identities, and hence electric and magnetic charges, are interchanged. This leads to a relation between the loop expansion parameter $g_d$ of the $(d-1)$-brane and the loop expansion parameter $g_{\tilde{d}}$ of the $(\tilde{d}-1)$-brane. We find

\[ g_d = 1/g_{\tilde{d}}, \quad (1.2) \]

thus confirming that strongly coupled $(d-1)$ branes correspond to weakly coupled $(\tilde{d}-1)$-branes and vice versa. The question of duality at higher orders in this loop expansion is considered in section 7 where we generalize the $D = 10$ string/fivebrane results of [15] to arbitrary $d$ and $\tilde{d}$.

Thus far, our discussion has been valid at arbitrary spacetime dimension $D$ and worldvolume dimension $d$. The important case of $D = 10$ is treated in section 8 where recover as special cases of our general solution the $N = 1$ superstring [10] and $N = 1$ superfivebrane [14], the Type IIA superparticle [2], superstring [18], supermembrane [2], superfourbrane [2], superfivebrane [18] and supersixbrane [2] and the Type IIB superstring [10], self-dual superthreebrane [22] and superfivebrane [18]. Another special case, the $D = 11$ supermembrane [12] is recovered in section 10, from which the $D = 10$ superstring follows by simultaneous dimensional reduction [29] of spacetime and worldvolume. $D = 6$ is of special interest because in this dimension a string is dual to another string. This could be either the usual strong/weak duality $\phi \rightarrow -\phi$ or else, in analogy with the threebrane in
$D = 10$, via a self-duality, and in section 11 we present a discussion of the $D = 6$ self-dual superstring.

In section 12 and 13, we turn to a case of obvious interest: $D = 4$. First of all in section 12, we recall the “electric” particle/“magnetic” monopole duality of Gibbons and Perry [30] and show how it follows a consequence of 0-brane/6-brane duality in $D = 10$. Secondly, in section 13, we recover another special case of our general solution the $D = 4$, $d = 0$ “axionic instanton” [31].

Solutions with $\mathcal{M}_d \geq \frac{1}{\sqrt{2}} g_{d\bar{d}}$ are discussed in section 14. These solutions exhibit singularities shielded by an event horizon. As special cases, we recover the $D = 10$ black $p$-branes $(p = 0, \ldots, 6)$ of Horowitz and Strominger [3], the $D = 11$ black $p$-branes $(p = 2, 5)$ of Gubser [6] and the $D = 4$ black hole $(p = 1)$ of Gibbons et al [30,32]. Finally in section 15, we generalize the results of [33] to show that although the elementary $(d - 1)$-brane is a singular solution of $(d - 1)$-brane theory, it is a non-singular soliton solution of $(d - 1)$-brane theory, and vice-versa.

2. General equations for arbitrary $(d, D)$

Consider an antisymmetric tensor potential of rank $d$, $A_{M_1 M_2 \ldots M_d}$, in $D$ spacetime dimensions ($M = 0, 1, \ldots (D - 1)$) interacting with gravity, $g_{MN}$, and the dilaton, $\phi$, via the action

$$I_D(d) = \frac{1}{2\kappa^2} \int d^Dx \sqrt{-g} \left( R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2(d + 1)!} e^{-\alpha(d)\phi} F_{d+1}^2 \right),$$

(2.1)

where the rank $(d + 1)$ field strength $F_{d+1}$ is given by

$$F_{d+1} = dA_d,$$

(2.2)

and $\alpha(d)$ is an, as yet undetermined, constant. Special cases of this action have been considered before in the context of classical solutions [34] [2-25]. Here we keep both $D$ and $d$ arbitrary. We allow these fields to couple to an elementary $d$-dimensional extended object (a “$(d - 1)$-brane”) whose trajectory is given by $X^M(\xi^i)$ ($i = 0, 1, \ldots (d - 1)$),
worldvolume metric by $\gamma_{ij}(\xi)$, and tension by $T_d$, via the action

\[
S_d = T_d \int d^d \xi \left( -\frac{1}{2} \sqrt{-\gamma} \gamma^{ij} \partial_i X^M \partial_j X^N g_{MN} e^{\alpha(d)\phi/d} + \frac{(d-2)}{2} \sqrt{-\gamma} \right. \\
\left. - \frac{1}{d!} \varepsilon_{i_1 i_2 \ldots i_d} \partial_{i_1} X^{M_1} \partial_{i_2} X^{M_2} \ldots \partial_{i_d} X^{M_d} A_{M_1 M_2 \ldots M_d} \right).
\] (2.3)

The $\phi$ dependence is chosen so that under the rescaling

\[
g_{MN} \to \lambda^{2d/(D-2)} g_{MN}, \\
A_{M_1 M_2 \ldots M_d} \to \lambda^d A_{M_1 M_2 \ldots M_d}, \\
e^\phi \to \lambda^{2(D-d-2)/(D-2)\alpha(d)} e^\phi, \\
\gamma_{ij} \to \lambda^2 \gamma_{ij},
\] (2.4)

both actions scale the same way

\[
I_D(d) \to \lambda^d I_D(d), \\
S_d \to \lambda^d S.
\] (2.5)

The field equations and Bianchi identities of the $A$ field may be written

\[
d^*(e^{-\alpha(d)\phi} F) = 2 \kappa^2 (-)^d * J, \] (2.6)

\[
dF \equiv 0, \] (2.7)

where the rank $d$ source $J$ is given by

\[
J^{M_1 \ldots M_d} = T_d \int d^d \xi \varepsilon^{i_1 i_2 \ldots i_d} \partial_{i_1} X^{M_1} \partial_{i_2} X^{M_2} \ldots \partial_{i_d} X^{M_d} \frac{\delta^D(x-X)}{\sqrt{-g}}. \] (2.8)

Let us introduce the dual worldvolume dimension, $\tilde{d}$, by

\[
\tilde{d} \equiv D - d - 2. \] (2.9)

We may now define two conserved charges: the Noether “electric” charge

\[
e_{d} = \frac{1}{\sqrt{2\kappa}} \int_{\Sigma^{d+1}} e^{-\alpha(d)\phi} F, \] (2.10)
where $S^{d+1}$ is the $(d + 1)$-sphere surrounding the elementary $(d - 1)$ brane, and the topological “magnetic” charge

$$g_{d} = \frac{1}{\sqrt{2\kappa}} \int_{S^{d+1}} F. \quad (2.11)$$

This latter charge will be non-zero if the action $I_D$ admits a solitonic $d$-dimensional extended object (a “$(d - 1)$-brane”). These charges obey a Dirac quantization condition [27,28],

$$\frac{\epsilon_d g_{d}}{4\pi} = \frac{n}{2}, \quad n = \text{integer} \quad (2.12)$$

analogous to the $(d = 1, D = 4)$ condition that relates electric and magnetic charges. At this stage, of course, it is not yet obvious that the system admits either elementary or solitonic extended object solutions, nor if they do, what are the values of the electric and magnetic charges $\epsilon_d$ and $g_d$.

Let us first consider the field equations resulting from $I_D + S_d$. The Einstein equation is

$$\sqrt{-g} \left[ R^{MN} - \frac{1}{2} g^{MN} R - \frac{1}{2} (\partial^M \phi \partial^N \phi) - \frac{1}{2} g^{MN} (\partial \phi)^2 \right]$$

$$- \frac{1}{2} \frac{1}{d!(F^{M_{1}\ldots M_{d}} F_{N_{1}\ldots N_{d}}) e^{-\alpha(d) \phi}}$$

$$= \kappa^2 \sqrt{-g} T^{MN} ((d - 1) - \text{brane}), \quad (2.13)$$

where the energy-momentum tensor is given by

$$T^{MN} ((d - 1) - \text{brane}) = -T_d \int d^d \xi \sqrt{-\gamma \gamma^{ij} \partial_i X^M \partial_j X^N e^{\alpha \phi/d} \frac{\delta^D(x - X)}{\sqrt{-g}}}, \quad (2.14)$$

the antisymmetric tensor equation is

$$\partial_M (\sqrt{-g} e^{-\alpha \phi} F^{M_{1}\ldots M_{d}}) = 2\kappa^2 T_d \int d^d \xi \varepsilon^{i_{1}\ldots i_{d}} \partial_{i_1} X^{M_{1}} \ldots \partial_{i_d} X^{M_{d}} \delta^D(x - X), \quad (2.15)$$

and the dilaton equation is

$$\partial_M (\sqrt{-g} g^{MN} \partial_N \phi) + \frac{\alpha(d)}{2(d + 1)!} \sqrt{-g} e^{-\alpha(d) \phi} F^2$$

$$= \frac{\alpha(d) \kappa^2 T_d}{d} \int d^d \xi \sqrt{-\gamma \gamma^{ij} \partial_i X^M \partial_j X^N g_{MN} e^{\alpha \phi/d} \delta^D(x - X)} \quad (2.16)$$
Furthermore, the \((d-1)\)-brane field equations are
\[
\partial_i (\sqrt{-\gamma} \gamma^{ij} \partial_j X^N g_{MN} e^\alpha (d) \phi / d) - \frac{1}{2} \sqrt{-\gamma} \gamma^{ij} \partial_i X^N \partial_j X^P \partial_M (g_{NP} e^\alpha (d) \phi / d)
- \frac{1}{d!} \varepsilon_{i_1 \ldots i_d} \partial_{i_1} X^{M_1} \ldots \partial_{i_d} X^{M_d} F_{M_1 \ldots M_d} = 0,
\]
and
\[
\gamma_{ij} = \partial_i X^M \partial_j X^N g_{MN} e^\alpha (d) \phi / d.
\]

3. The elementary \((d-1)\)-brane

To solve these coupled field-(\(d-1\))-brane equations we begin by making an ansatz for the \(D\)-dimensional metric \(g_{MN}\), \(d\)-form \(A_{M_1 \ldots M_d}\), dilaton \(\phi\) and coordinates \(X^M(\xi)\) corresponding to the most general \(d/(D-d)\) split invariant under \(P_d \times SO(D-d)\) where \(P_d\) is the \(d\)-dimensional Poincaré group. We split the indices
\[
x^M = (x^\mu, y^m),
\]
where \(\mu = 0, 1, \ldots (d-1)\) and \(m = d, d+1, \ldots (D-1)\), and write the line-element as
\[
ds^2 = e^{2A} \eta_{\mu \nu} dx^\mu dx^\nu + e^{2B} \delta_{mn} dy^m dy^n,
\]
and the \(d\)-form gauge field as
\[
A_{\mu_1 \ldots \mu_d} = -\frac{1}{d!} g^{\varepsilon_{\mu_1 \ldots \mu_d}} e^C,
\]
where \(^d g\) is the determinant of \(g_{\mu \nu}\), \(\varepsilon_{\mu_1 \ldots \mu_d} \equiv g_{\mu_1 \nu_1} \ldots g_{\mu_d \nu_d} \varepsilon^{\nu_1 \ldots \nu_d}\) and \(\varepsilon^{012 \ldots (d-1)} = 1\) i.e. \(A_{01 \ldots (d-1)} = -e^C\). All other components of \(A_{M_1 \ldots M_d}\) are set to zero. \(P_d\) invariance requires that the arbitrary functions \(A, B, C\) depend only on \(y^m\); \(SO(D-d)\) invariance then requires that this dependence be only through \(y = \sqrt{\delta_{mn} y^m y^m}\). Similarly our ansatz for the dilaton is
\[
\phi = \phi(y).
\]
In the \((d-1)\)-brane sector we also split
\[
X^M = (X^\mu, Y^m),
\]
and make the static gauge choice

$$X^\mu = \xi^\mu,$$

and the ansatz

$$Y_m = \text{constant}.$$ (3.7)

Substituting these ansatz into (2.18) yields

$$\gamma_{ij} = e^{2A + \alpha(d)\phi/\xi} \eta_{ij},$$ (3.8)

and the only non-vanishing components of the field strength are

$$F_{m\mu_1 \ldots \mu_d} = -\frac{1}{g} \tilde{e}_{\mu_1 \ldots \mu_d} \partial_m e^C.$$ (3.9)

Then the $\mu\nu$ components of the Einstein equation (2.13) reduce to a single equation

$$e^{(d-2)A + dB} \delta^{mn} \left[ (d-1)\partial_m \partial_n A + \frac{d(d-1)}{2} \partial_m A \partial_n A + (\tilde{d} + 1) \partial_m \partial_n B \right. \\
+ \frac{(\tilde{d} + 1)d}{2} \partial_m B \partial_n B + \tilde{d}(d-1) \partial_m A \partial_n B \\
+ \frac{1}{4} e^{-2A + 2C - \alpha(d)\phi} \partial_m C \partial_n C + \frac{1}{4} \partial_m \phi \partial_n \phi \right] \\
= -\kappa^2 T_d e^{(d-2)A + \alpha(d)\phi/2} \delta^{D-d}(y),$$ (3.10)

and the $mn$ components reduce to

$$e^{dA + (d-2)B} \left[-\tilde{d}\partial^m \partial^n B + \delta^{mn} \tilde{d} \delta^{kl} \partial_k \partial_l B \\
- d\partial^m \partial^n A + d\delta^{mn} \delta^{kl} \partial_k \partial_l A - d\partial^m A \partial^n A + \frac{d(d+1)}{2} \delta^{mn} \delta^{kl} \partial_k A \partial_l A \\
+ d(\partial^m A \partial^n B + \partial^m B \partial^n A + (\tilde{d} - 1) \delta^{mn} \delta^{kl} \partial_k A \partial_l B) \\
- \frac{1}{2} \partial^m \phi \partial^n \phi + \frac{1}{4} \delta^{mn} \delta^{kl} \partial_k \phi \partial_l \phi \right] \\
- \frac{1}{2} e^{-dA + (d-2)B + 2C - \alpha(d)\phi} \left[-\partial^m C \partial^n C + \frac{1}{2} \delta^{mn} \delta^{kl} \partial_k C \partial_l C \right] \\
= 0.$$ (3.11)

The antisymmetric tensor field equation (2.15) becomes

$$\delta^{mn} \partial_m \left[e^{-\alpha(d)\phi-dA + dB} \partial_n e^C \right] = 2\kappa^2 T_d \delta^{D-d}(y),$$ (3.12)
and the dilaton equation (2.16) becomes
\[
\delta^{mn} \partial_m \left( e^{dA + \hat{d}B} \partial_n \phi \right) - \frac{\alpha(d)}{2} e^{-dA + \hat{d}B + 2C - \alpha(d)\phi} \delta^{mn} \partial_m C \partial_n C = \alpha(d)\kappa^2 T_d e^{dA + \alpha(d)\phi/2} e^{(D-d)(y)}.
\]
(3.13)
Finally, the \((d-1)\)-brane equation (2.17) becomes
\[
\partial_m \left( e^{dA + \alpha(d)\phi/2} = e^C \right) = 0.
\]
(3.14)
Hence we have five equations for the four unknown functions \(A, B, C, \phi\) and the unknown parameter \(\alpha(d)\).

The unique solution, assuming that \(g_{MN}\) tends asymptotically to \(\eta_{MN}\), is given by
\[
A = \frac{\tilde{d}}{2(d + \hat{d})} (C - C_o),
\]
\[
B = -\frac{\tilde{d}}{2(d + \hat{d})} (C - C_o),
\]
\[
\frac{\alpha(d)}{2} \phi = \frac{\alpha^2(d)}{4} (C - C_o) + C_o,
\]
where \(C_o = \alpha\phi_o/2\) and \(\phi_o\) is the dilaton vev. \(C\) is given by
\[
e^{-C} = e^{-C_o} + \frac{k_d}{y^\hat{d}}, \quad \hat{d} > 0
\]
(3.16)
\[
e^{-C} = e^{-C_o} - \frac{\kappa^2 T_d}{\pi} \ln y, \quad \hat{d} = 0
\]
and
\[
k_d = 2\kappa^2 T_d / \hat{d} \Omega_{\hat{d}+1},
\]
(3.17)
where \(\Omega_{\hat{d}+1}\) is the volume of \(S^{\hat{d}+1}\). The parameter \(\alpha(d)\) is given by
\[
\alpha^2(d) = 4 - \frac{2dd}{d + \hat{d}}.
\]
(3.18)
Note, incidentally, that for these solutions, the coefficients of the \(\delta\)-function in (3.10) and (3.13) vanish at \(y = 0\). So the Einstein equation and the dilaton equation are essentially source-free; only in the antisymmetric tensor equation is a \(\delta\)-function source. We shall return to this in section 5.

A crucial result of this section is that we have fixed the constant \(\alpha(d)\) as in (3.18) by the requirement that our theory (2.1) yield elementary \((d-1)\)-brane solutions.
4. Bogomol’nyi bounds and the “no-force” condition

The mass per unit \((d - 1)\)-volume of the elementary \((d - 1)\)-brane is given by

\[
\mathcal{M}_d = \int d^{D-d} y \theta_{oo},
\]

where \(\theta_{MN}\) is the total energy-momentum pseudotensor of the combined gravity-matter system. One may generalize the \(d = 2\), \(\phi_o = 0\), arguments of Dabholkar et al [10] to arbitrary \(d\) and non-vanishing \(\phi_o\) and establish a Bogomol’nyi bound

\[
\kappa \mathcal{M}_d \geq \frac{1}{\sqrt{2}} | \epsilon_d | e^{C_o} = \frac{1}{\sqrt{2}} | \epsilon_d | e^{\alpha(d) \phi_o / 2},
\]

where \(\epsilon_d\) is the electric charge of (2.10). For the solution of Section 3, we find

\[
\mathcal{M}_d = T_d e^{C_o}.
\]

To compute \(\epsilon_d\) it is convenient to introduce polar coordinates

\[
y^m = (y, \theta^i),
\]

where \(i = 1, \ldots, (d + 1)\), so that

\[
\delta_{mn} dy^m dy^n = dy^2 + y^2 d\Omega^2_{d+1},
\]

where \(d\Omega^2_{d+1}\) is the metric on the unit \(S^{d+1}\). Then we note from (3.9) that

\[
F_\mu_1 \cdots \mu_d = -\frac{1}{g} \varepsilon_{\mu_1 \cdots \mu_d} \partial_y e^C,
\]

The dual of \(F, F\), has non-vanishing components only in the \(\theta^i\) directions

\[
\sqrt{-g} F^{\theta_1 \cdots \theta_{d-d-1}} = \left(-\right)^{(D-d)(d+1)} e^{2C} \partial_y e^{-C},
\]

Hence, using (3.15–18) we find

\[
e^{-\alpha \phi^*} F_{\theta_1 \cdots \theta_{D-d-1}} = \left(-\right)^{(D-d)(d+1)} 2\kappa^2 T_d \frac{\varepsilon_{\theta_1 \cdots \theta_{D-d-1}}}{\Omega_{d+1}}.
\]
It follows from (2.10) that
\[ c_d = \sqrt{2} \kappa T_d (-)^{(D-d)(d+1)}, \]
and hence
\[ \mathcal{M}_d = \frac{1}{\sqrt{2}} \left| c_d \right| e^{\alpha(d) \phi_c / 2}, \]
and the bound in (4.2) is saturated. This shows that these elementary \((d-1)\)-brane solutions are stable.

So far we have concentrated on single \((d-1)\)-brane solutions of the field equations. However, there is a straightforward generalization to exact, stable multi-\((d-1)\)-brane configurations obtained by a linear superposition of the solutions (3.16),
\[ \epsilon^{-C} = \epsilon^{-C_0} + \sum_i \frac{k_d}{\left| y - y_i \right|^2}, \]
where \(y_i\) corresponds to the position of each \((d-1)\)-brane. The ability to superpose solutions of this kind is a well-known phenomenon in soliton and instanton physics and goes by the name of the “no-force condition” [10]. In the present context, it means that the mutual gravitational-dilaton attraction of two separated \((d-1)\)-branes is exactly cancelled by an equal and opposite contribution from the antisymmetric tensor. To see this explicitly, consider the multi-\((d-1)\)-brane configuration (4.11) with, for example, \(n\) \((d-1)\)-branes as sources. In general, we do not have the transverse \(SO(D-d)\) symmetry, but we still have the \(P_d\) Poincare symmetry for the configuration (4.11). Let each \((d-1)\)-brane with label \(i\) satisfy \(X^\mu(i) = \xi^\mu\) so that, in particular, they all have the same orientation. The Lagrangian for each of the \((d-1)\)-branes with label \(i\) in the fields of the sources given by (3.1–4) is, from (2.3)
\[ \mathcal{L}_d = -T_d \left[ \sqrt{-dt(e^{2A+\alpha(d)\phi/d}\eta_{ij} + e^{2B+\alpha(d)\phi/d}\partial_i Y^m(l)\partial_j Y_m(l) - \epsilon^C)} \right] \]
corresponding to a potential
\[ V = T_d(\epsilon^{dA+\alpha(d)\phi/2} - \epsilon^C), \]
but this vanishes by (3.14). This generalizes to arbitrary \(d\) and \(D\) the “no-force condition” for strings [10], fivebranes [14] in \(D = 10\) and membranes in \(D = 11\) [12]. Expanding out
we find
\[ \mathcal{L} = -\frac{T_d}{2} \epsilon^{(d-2)A+2B+\alpha(d)\phi/2}\eta_{ij}\partial_i Y^m \partial_j Y_m + \ldots, \] (4.14)
and so the absence of velocity-dependent forces corresponds to
\[ (d-2)A + 2B + \alpha(d)\phi/2 = \text{constant}, \] (4.15)
which is indeed satisfied by virtue of (3.15) and we find that the constant is just \( C_0 \). This generalizes to arbitrary \( d \) and \( D \), the absence of velocity dependent forces for strings and fivebranes in \( D = 10 \) [35].

5. The solitonic \((\tilde{d} - 1)\)-brane

The elementary \((\tilde{d} - 1)\)-branes we have discussed so far correspond to solutions of the coupled field-brane system with action \( I_D(\tilde{d}) + S_d \). As such they exhibit \( \delta \)-function singularities at \( y = 0 \). They are characterized by a non-vanishing Noether “electric” charge \( \epsilon_d \). By contrast, we now wish to find solitonic \((\tilde{d} - 1)\)-brane, corresponding to solutions of the source free equations resulting from \( I_D(\tilde{d}) \) alone, which are regular at \( y = 0 \), and will be characterized by a non-vanishing topological “magnetic” charge \( g_{\tilde{d}} \). (Recall that \( \tilde{d} = D - d - 2 \)).

To this end, we now make an ansatz invariant under \( P_{\tilde{d}} \times SO(D - \tilde{d}) \). Hence we write (3.1) and (3.2) as before where now \( \mu = 0,1 \ldots (\tilde{d} - 1) \) and \( m = \tilde{d}, \tilde{d} + 1, \ldots (D - 1) \).

The ansatz for the antisymmetric tensor, however, will now be made on the field strength rather than on the potential. From section 3 we recall that a non-vanishing electric charge corresponds to
\[ \frac{1}{\sqrt{2\kappa}} e^{-\alpha\phi} F_{\tilde{d}+1} = \epsilon_d \bar{\epsilon}_{\tilde{d}+1} / \Omega_{\tilde{d}+1}, \] (5.1)
where \( \bar{\epsilon}_{\tilde{d}+1} \) is the volume form on \( S_{\tilde{d}+1} \). Accordingly, to obtain a non-vanishing magnetic charge, we make the ansatz
\[ \frac{1}{\sqrt{2\kappa}} F_{\tilde{d}+1} = g_{\tilde{d}} \bar{\epsilon}_{\tilde{d}+1} / \Omega_{\tilde{d}+1}, \] (5.2)
where \( \bar{\epsilon}_{\tilde{d}+1} \) is the volume form on \( S_{\tilde{d}+1} \). Since this is an harmonic form, \( F \) can no longer be written globally as the curl of \( A \), but it satisfies the Bianchi identities. It is now not
difficult to show that all the field equations are satisfied simply by making the replacement $d \to \tilde{d}$, and hence $\alpha(d) \to \alpha(\tilde{d}) = -\alpha(d)$ in (3.15–18). For future reference we write the explicit solution in the case $\phi_0 = 0$

$$ds^2 = \left(1 + \frac{k_{\tilde{d}}}{y^{\tilde{d}}}\right)^{-d/(d+\tilde{d})} dx^\mu dx_\mu + \left(1 + \frac{k_{\tilde{d}}}{y^{\tilde{d}}}\right)^{\tilde{d}/(d+\tilde{d})} dy_\mu dy_\mu,$$

$$e^{2\phi} = \left(1 + \frac{k_{\tilde{d}}}{y^{\tilde{d}}}\right)^{\alpha(d)},$$

$$F_{d+1} = \sqrt{2\kappa g_{\tilde{d}} \tilde{\varepsilon}_{d+1}/\Omega_{d+1}}.$$ 

Note that by this device, we have found solutions everywhere including $y = 0$, since the $\delta$-functions were already absent in the Einstein and dilaton equations.

It follows that the mass per unit $(\tilde{d} - 1)$-volume now saturates a bound involving the magnetic charge

$$\mathcal{M}_{\tilde{d}} = \frac{1}{\sqrt{2}} \left| g_{\tilde{d}} \right| e^{\alpha(d) \phi_0/2}$$

$$= \frac{1}{\sqrt{2}} \left| g_{\tilde{d}} \right| e^{-\alpha(d) \phi_0/2}. \quad (5.4)$$

Note that the $\phi_0$ dependence is such that $\mathcal{M}_{\tilde{d}}$ is large for small $\mathcal{M}_d$ and vice-versa.

The electric charge of the elementary solution and the magnetic charge of the soliton solution obey a Dirac quantization rule [27,28]

$$e_d g_{\tilde{d}} = 2\pi n, \quad n = \text{integer}, \quad (5.5)$$

and hence from (4.9)

$$(-)^{(D-d)(d+1)} g_{\tilde{d}} = 2\pi n/\sqrt{2\kappa T_{\tilde{d}}}, \quad (5.6)$$

6. Duality

We now wish to consider the theory “dual” to (2.1) for which the roles of field equations (2.6) and Bianchi identities (2.7) are interchanged. To this end let us write the action

$$\tilde{I}_D(\tilde{d}) = \frac{1}{2\kappa^2} \int d^D x \sqrt{-g} \left( R - \frac{1}{2} \left( \partial \phi \right)^2 - \frac{1}{2(d+1)!} e^{\alpha(d) \phi} \tilde{F}_{d+1}^2 \right), \quad (6.1)$$
where the rank $(\tilde{d} + 1)$ field strength $\tilde{F}$ is given by

$$\tilde{F}_{\tilde{d} + 1} = d\tilde{A}_{\tilde{d}}.$$  

(6.2)

$\alpha(d)$ is the same constant as appearing in (2.1) but appears with opposite sign, i.e.

$$\alpha(\tilde{d}) = -\alpha(d).$$  

(6.3)

Allow these fields to couple to an elementary $\tilde{d}$-dimensional extended object (a “$(\tilde{d} - 1)$-brane”) with action

$$\tilde{S}_\tilde{d} = T_\tilde{d} \int d^\tilde{d} \xi \left( -\frac{1}{2} \sqrt{-\gamma} \gamma^{ij} \partial_i X^M \partial_j X^N g_{MN} e^{-\alpha(\tilde{d})\phi/\tilde{d}} + \frac{(\tilde{d} - 2)}{2} \sqrt{-\gamma} \right. $$

$$- \frac{1}{d!} \epsilon^{i_1 i_2 \ldots i_d} \partial_i X^{M_1} \partial_{i_2} X^{M_2} \ldots \partial_{i_d} X^{M_d} A_{M_1 M_2 \ldots M_d} \left. \right).$$

(6.4)

The $\phi$ dependence is such that under the rescaling

$$g_{MN} \to \tilde{\lambda}^{2\tilde{d}/(D-2)} g_{MN},$$

$$\tilde{A}_{M_1 \ldots M_\tilde{d}} \to \tilde{\lambda}^d \tilde{A}_{M_1 \ldots M_\tilde{d}},$$

$$e^\phi \to \tilde{\lambda}^{-2\tilde{d}(D-\tilde{d}-2)/(D-2)\alpha(\tilde{d})} e^\phi,$$

$$\gamma_{ij} \to \tilde{\lambda}^2 \gamma_{ij},$$

(6.5)

both actions scale the same way

$$\tilde{I}_D(\tilde{d}) \to \tilde{\lambda}^{\tilde{d}} I_D(d),$$

$$\tilde{S}_\tilde{d} \to \tilde{\lambda}^{\tilde{d}} \tilde{S}_\tilde{d}.$$  

(6.6)

The field equations and Bianchi identities of the $\tilde{A}$ field may be written

$$d^* (e^{\alpha(\tilde{d})\phi} \tilde{F}) = 2\kappa^2 (-)^{\tilde{d}^2} \tilde{j},$$

$$d\tilde{F} = 0.$$  

(6.7)

(6.8)

It should be clear that the system described by $\tilde{I}_D(\tilde{d}) + \tilde{S}_\tilde{d}$ admit the same elementary solutions as that described by $I_D(d) + S_d$ and that $\tilde{I}_D(\tilde{d})$ alone admits the same solitonic
solutions as $I_D(d)$ alone, provided we everywhere make the replacement $d \to \tilde{d}$ and hence $\alpha(d) \to \alpha(\tilde{d}) = -\alpha(d)$. In particular the Noether electric charge is given by

$$\bar{e}_d = \frac{1}{\sqrt{2\kappa}} \int_{S^{d+1}} e^{\alpha \phi} \tilde{F}_{d+1},$$

(6.9)

and the topological magnetic charge by

$$\bar{g}_d = \frac{1}{\sqrt{2\kappa}} \int_{S^{d+1}} \tilde{F}_{d+1},$$

(6.10)

and they obey the condition

$$\bar{e}_d \bar{g}_d = 2\pi n.$$

(6.11)

So far we have discovered that the equations of $I_D(d)$ admit an elementary $(d - 1)$-brane solution and a solitonic $(\tilde{d} - 1)$-brane solution. Conversely, the equations of $\tilde{I}_D(\tilde{d})$ admit an elementary $(\tilde{d} - 1)$-brane solution and a solitonic $(d - 1)$-brane solution. We now wish to go a step further and assert that the $(d - 1)$ brane is “dual” to the $(\tilde{d} - 1)$-brane. In its strongest sense this means that the two theories are equivalent descriptions of the same physics. In the present context, however, we simply make the assumption that the $I_D(d)$ and $\tilde{I}_D(\tilde{d})$ are equivalent i.e we assume that the metric $g_{MN}$ and dilaton $\phi$ are the same and that the $(\tilde{d} + 1)$-form field strength $\tilde{F}_{\tilde{d} + 1}$ is dual to the $(d + 1)$-form field strength $F_{d+1}$. More precisely,

$$\tilde{F}_{\tilde{d} + 1} = e^{-\alpha(d) \phi} F_{d+1}$$

(6.12)

so that the (source-free) field equations and Bianchi identities of $I_D(d)$, (2.6) and (2.7), become the Bianchi identities and (source-free) field equations of $\tilde{I}_D(\tilde{d})$, (6.8) and (6.7). This leads immediately to

$$e_d = g_d,$$

(6.13)

$$g_{\tilde{d}} = \bar{e}_d,$$

and hence

$$\kappa^2 T_d T_{\tilde{d}} = |n| \pi$$

(6.14)

The duality assumption also leads to a relation between the dimensionless loop expansion parameters of the $(d - 1)$-brane and the $(\tilde{d} - 1)$-brane. To see this we note that
metrics appearing naturally in \((d - 1)\)-brane and \((\tilde{d} - 1)\)-brane \(\sigma\)-models (2.3) and (6.4) are
\[
g_{MN}(d) = e^{\alpha(d)\phi/d} g_{MN}(\text{canonical}), \tag{6.15}
g_{MN}(\tilde{d}) = e^{-\alpha(d)\phi/\tilde{d}} g_{MN}(\text{canonical}). \tag{6.16}
\]

If we rewrite \(I_D(d)\) and \(\tilde{I}_D(\tilde{d})\) in these variables we find
\[
I_D(d) = \frac{1}{2\kappa^2} \int d^Dx \sqrt{-g} e^{-(D-2)\alpha(d)\phi/2d} \left[ R - \frac{1}{2} \left( 1 - \frac{\alpha^2(D - 1)(D - 2)}{2d^2} \right)(\partial\phi)^2 \right.
\]
\[
- \frac{1}{2} \left. \frac{1}{2(d + 1)!} F_{d+1}^2 \right], \tag{6.17}
\]
and
\[
\tilde{I}_D(\tilde{d}) = \frac{1}{2\kappa^2} \int d^Dx \sqrt{-g} e^{(D-2)\alpha(d)\phi/2\tilde{d}} \left[ R - \frac{1}{2} \left( 1 - \frac{\alpha^2(D - 1)(D - 2)}{2\tilde{d}^2} \right)(\partial\phi)^2 \right.
\]
\[
- \frac{1}{2} \left. \frac{1}{2(\tilde{d} + 1)!} F_{\tilde{d}+1}^2 \right]. \tag{6.18}
\]

Note that in both cases a common dilaton-dependent factor appears. This reveals that the \((d - 1)\)-brane loop counting parameter is
\[
g_d = e^{(D-2)\alpha(d)\phi/4d}, \tag{6.19}
\]
and the \((\tilde{d} - 1)\)-brane loop counting parameter is
\[
g_{\tilde{d}} = e^{-(D-2)\alpha(d)\phi/4\tilde{d}}. \tag{6.20}
\]

Hence
\[
g^d_d = 1/g_{\tilde{d}}^{\tilde{d}}, \tag{6.21}
\]
and strongly coupled \((d - 1)\) branes correspond to weakly coupled \((\tilde{d} - 1)\) branes and vice-versa.

Finally we note that, in the case of \(d = 2\), the following field redefinition
\[
(D - 2)\alpha(2)\phi = 8\Phi \tag{6.22}
\]

yields from (6.17) an \( I_D(d) \) which is \( D \)-independent, namely
\[
I_D(2) = \frac{1}{2\kappa^2} \int d^Dx \sqrt{-g} e^{-2\Phi} \left[ R + 4(\partial \Phi)^2 - \frac{1}{2.3!} F_3^2 \right].
\] (6.23)

This is a well-known result in string theory. Curiously, there is no field redefinition which renders \( I_D(d) \) independent of \( D \) for \( d \neq 2 \). However, we may dualize (6.23) to obtain
\[
\tilde{I}_D(D-4) = \frac{1}{2\kappa^2} \int d^Dx \sqrt{-g} \ e^{a\Phi/D-4} \left[ R - \frac{4(D-10)}{(D-4)^2} (\partial \Phi)^2 - \frac{1}{2(D-3)!} \tilde{F}_{D-3}^2 \right].
\] (6.24)

In these string variables the metric of the elementary string is given by
\[
ds^2 = \left(1 + \frac{k_2 e^{C_o}}{y^{D-4}}\right)^{-1} \eta_{\mu\nu} dx^\mu dx^\nu + \delta_{mn} dy^m dy^n
\] (6.25)

with \( \mu = 0,1 \) and \( m = 1 \ldots D - 2 \). Also
\[
\alpha(2) = \sqrt{\frac{8}{D-2}},
\] (6.26)

so
\[
\Phi = \frac{1}{2}(C - C_o) + \frac{D-2}{4} C_o,
\] (6.27)

where
\[
e^{-C} = e^{-C_o} + \frac{k_2}{y^{D-4}}, \quad D > 4
\]
\[
e^{-C} = e^{-C_o} - \frac{k_2 T_2}{\pi} \ln y, \quad D = 4
\] (6.28)

On the other hand the solitonic \((D - 5)\)-brane is given by
\[
ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{k_{D-4}}{y^2} e^{C_o}\right) \delta_{mn} dy^m dy^n
\] (6.29)

where \( \mu = 0 \ldots D - 5 \) and \( m = D - 4, \ldots, D - 1 \). Also
\[
\alpha(D - 4) = -\sqrt{\frac{8}{D-2}},
\] (6.30)

so
\[
\Phi = -\frac{1}{2} (C - C_o) - \frac{(D-2)}{4} C_o,
\] (6.31)

where
\[
e^{-C} = e^{-C_o} + \frac{k_{D-4}}{y^2}.
\] (6.32)

We note that in these string \( \sigma \)-model variables the transverse part of the metric in (6.25) is flat and the spacetime part of the metric in (6.29) is flat. These are therefore free field theories from the point of view of conformal field theory.
7. Higher loops

$D$-dimensional strings involve two kinds of loop expansion: quantum $D = 10$ strings loops ($L$) with loop expansion parameter $\frac{\kappa^2 e^{2\phi}}{(2\pi)^{D/2}}$ and classical $d = 2 \sigma$-model loops with loop expansion parameter $\alpha'_2 \equiv 1/2\pi T_2$, assuming we use the string $\sigma$-model metric. Similarly a $D$-dimensional $(d-1)$-brane will presumably require quantum $D$-dimensional $(d-1)$-brane loops ($L$) with loop expansion parameter $\frac{\kappa^2 e^{(D-2)\alpha(d)\phi/2d}}{(2\pi)^{D/2}}$ and classical $d$-dimensional $\sigma$-model loops with loop expansion parameter $\alpha'_d \equiv 1/(2\pi)^{d/2} T_d$. Let us consider the purely gravitational contribution to the resulting effective action, using the $(d-1)$-brane $\sigma$-model metric:

$$\mathcal{L}_{LL+m} = a_{LL+m} \frac{1}{2\kappa^2} \sqrt{-g} e^{-(D-2)\alpha(d)\phi/2d} \left( \frac{2\kappa^2 e^{(D-2)\alpha(d)\phi/2d}}{(2\pi)^{D/2}} \right)^L \alpha'_d \, R^m, \quad (7.1)$$

where $R^n$ is symbolic for a scalar contribution of $n$ Riemann tensors each of dimension 2. One could also include covariant derivatives of $R$ but, for our purposes (7.1) will be sufficient. The $a_{LL+m}$ are numerical coefficients, not involving $\pi$. Since

$$[\mathcal{L}_{LL+m}] = D, \quad [\kappa^2] = 2 - D, \quad [\alpha'_d] = -d, \quad (7.2)$$

we have, on dimensional grounds,

$$dm = 2(n - 1) - (D - 2)L. \quad (7.3)$$

By the same argument, a $D$-dimensional ($\tilde{d} - 1$)-brane will require quantum $D$-dimensional ($\tilde{d} - 1$)-brane loops ($\tilde{L}$) with loop expansion parameter $\frac{\kappa^2 e^{-(D-2)\alpha(d)\phi/2\tilde{d}}}{(2\pi)^{D/2}}$ and classical $\tilde{d}$-dimensional $\sigma$-model loops with loop expansion parameter $\alpha'_{\tilde{d}} \equiv 1/(2\pi)^{\tilde{d}/2} T_{\tilde{d}}$. The corresponding Lagrangian using the ($\tilde{d} - 1$)-brane $\sigma$-model metric is

$$\tilde{\mathcal{L}}_{\tilde{L}+\tilde{m}\tilde{L}} = \tilde{a}_{\tilde{L}+\tilde{m}\tilde{L}} \frac{1}{2\kappa^2} \sqrt{-g} e^{-(D-2)\alpha(d)\phi/2\tilde{d}} \left( \frac{2\kappa^2 e^{-(D-2)\alpha(d)\phi/2\tilde{d}}}{(2\pi)^{D/2}} \right)^{\tilde{L}} \alpha'_{\tilde{d}} \, R^m. \quad (7.4)$$

Again, on dimensional grounds,

$$\tilde{d}m = 2(n - 1) - (D - 2)\tilde{L}. \quad (7.5)$$
Our fundamental assumption is that $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are related by duality which implies, in particular, that the purely gravitational contributions should be identical when written in the same variables. So transforming to the canonical metric using (6.15) and (6.16), we find

\[
\mathcal{L}_{LL+m} = \frac{1}{2\kappa^2} a_{LL+m} \left( \frac{2\kappa^2}{(2\pi)^{D/2}} \right)^L a_m' e^{-\alpha(d) m \phi / 2} \sqrt{-g} R^n, \tag{7.6}
\]

\[
\tilde{\mathcal{L}}_{\tilde{L}+\tilde{m}L} = \frac{1}{2\kappa^2} \tilde{a}_{\tilde{L}+\tilde{m}L} \left( \frac{2\kappa^2}{(2\pi)^{D/2}} \right)^{\tilde{L}} \tilde{a}_m' \tilde{e}^{\alpha(\tilde{d}) \tilde{m} \phi / 2} \sqrt{-g} R^n, \tag{7.7}
\]

where we have dropped the terms like $(\partial \phi)^2 m R^{n-m}$ for $m = 1, \ldots, n$. Bearing in mind that from (6.14) with unit integer,

\[
2\kappa^2 = (2\pi)^{D/2} a_d' a_d', \tag{7.8}
\]

we find that $\mathcal{L}$ and $\tilde{\mathcal{L}}$ do coincide provided

\[
m + \tilde{m} = 0, \tag{7.9}
\]

i.e from (7.3) and (7.5), provided

\[
m = \tilde{L} - L = -\tilde{m}, \tag{7.10}
\]

\[
2n = \tilde{d}L + d\tilde{L} + 2, \tag{7.11}
\]

with

\[
a_{LL} = (-)^{(L-\tilde{L})D} \tilde{a}_{\tilde{L}\tilde{L}}, \tag{7.12}
\]

and hence

\[
\mathcal{L}_{LL} = a_{LL} (-)^{DL} \frac{1}{2\kappa^2} \alpha_d' \alpha_d e^{\alpha(L-\tilde{L}) \phi / 2} \sqrt{-g} R^{(\tilde{d}L+d\tilde{L}+2)/2}. \tag{7.13}
\]

This generalizes the result of [15], where (7.13) was obtained for $d = 2, \tilde{d} = 6$. Interestingly, it was there obtained in the context of the heterotic string, but here we see that the result is, in fact, universal. A similar analysis for the pure antisymmetric tensor terms yields, in the canonical metric,

\[
\mathcal{L}_{LL+m} = \frac{1}{2\kappa^2} a_{LL+m} \left( \frac{2\kappa^2}{(2\pi)^{D/2}} \right)^L a_m' e^{-\alpha(d) m \phi / 2} \sqrt{-g} \left( \frac{1}{2 \cdot (d+1)!} F_{d+1}^d e^{-\alpha(d) \phi} \right)^n, \tag{7.14}
\]
\[ \mathcal{L}_{L+\tilde{m}L} = \frac{1}{2\kappa^2} a_{L+\tilde{m}L} \left( \frac{2\kappa^2}{(2\pi)^{D/2}} \right)^L a_d \tilde{m} e^{\alpha(d)\tilde{m}\phi/2} \sqrt{-g} \left( \frac{1}{2 \cdot (d+1)!} F_{d+1}^2 e^{\alpha(d)\phi} \right)^n. \] 

(7.15)

and again we find from (6.12) that the Bianchi identities and antisymmetric tensor field equations of \( \mathcal{L} \) and \( \tilde{\mathcal{L}} \) are interchanged provided (7.9–12) are satisfied.

We note that under the rescalings (2.4) and (6.5) \( \mathcal{L}_{L\tilde{L}} \) scales as

\[ \mathcal{L}_{L\tilde{L}} \to \lambda^{d(1-L)} \tilde{\lambda}^{(1-L)} \mathcal{L}_{L\tilde{L}}. \]

(7.16)

As in [15], (7.13) gives rise to an infinite number of non-renormalization theorems. The first of which is the absence of a cosmological term \( \sqrt{-g} R^0 \), assuming that the total Lagrangian is given by

\[ \mathcal{L} = \sum_{L=0}^{\infty} \sum_{\tilde{L}=0}^{\infty} \mathcal{L}_{L\tilde{L}}. \]

(7.17)

The second states that \( \sqrt{-g} R \) appears only at \( (L = 0, \tilde{L} = 0) \) and hence the tree level action (2.1) does not get renormalized.

All this assumes, of course, that both the \( (d-1) \)-brane and the \( (\tilde{d}-1) \)-brane are quantum mechanically consistent. This will not be true in general but only some specific choices of \( d \) and \( D \). We intend to return to the question of quantum consistency elsewhere.

8. \( D = 10 \)

So far, our analysis has kept both the dimension of the worldvolume, \( d \), and the dimension of spacetime, \( D \), arbitrary. However, in the case of strings \( (d = 2) \) we know that certain spacetime dimensions are singled out for special attention. For example, Green-Schwarz superstrings exist classically only for \( D = 3, 4, 6 \) and 10 and, of these, only the \( D = 10 \) string is allowed quantum-mechanically in the sense of being anomaly-free [36]. (By the way, “\( D = 10 \)” is a loose way of speaking about central charge \( c = 15 \), so it could equally well mean a lower dimensional string with the correct amount of internal degrees of freedom.) Similarly, the critical dimension of the bosonic string is \( D = 26 \). Thus from our general discussion of sections 3 and 4, we see that, in addition to the elementary string solution [10], the bosonic string in \( D = 26 \) also admits a solitonic 21-brane solution.
In the case of \( N = 1 \) super \((d - 1)\)-branes, the classically allowed supersymmetric extended objects have been classified by Achucarro et al. [37] and correspond to the circles on the “brane-scan” of [2]. However, this classification is inadequate for \( N = 2 \) super \((d - 1)\)-branes when \( d > 2 \) since such Type II supersymmetric extended objects require spin \( > 1/2 \) fields on the worldvolume [18,19], which were excluded by the assumption in [37]. We have shown elsewhere [2] that the \((d-1)\)-brane solutions of sections 2 and 3 provide solutions of Type IIA supergravity in \( D = 10 \) for \( d = 1, 2, 3, 5, 6, 7 \) (i.e \( \tilde{d} = 7, 6, 5, 3, 2, 1 \)) only, and of Type IIB supergravity in \( D = 10 \) for \( d = 2, 4, 6 \) (i.e \( \tilde{d} = 6, 4, 2 \)) only. The existence of the Type IIA and IIB superstring solutions was established by Dabholkar et al [10], the Type IIA and IIB superfivebrane solutions by Callan et al [18], and the self-dual Type IIB superthreebrane by the present authors [22]. Now Horowitz and Strominger [3] have exhibited a two-parameter family of solutions of \( D = 10 \) Type IIA and B supergravity with event horizons: for \( d = 1, 2, 3, 4, 5, 6, 7 \) “black \((d - 1)\)-branes”. In some respects, these solutions resemble the Reissner-Nordstrom black-hole solution of general relativity which is known to admit unbroken supersymmetry in the extreme charge = mass limit. Horowitz and Strominger then conjectured that, in this limit, their black p-branes would also be supersymmetric and hence that there exist Type II super \((d - 1)\)-branes for all these values of \( d \). As we have shown in [2], this is indeed the case.

For \( D = 10 \), \( \tilde{d} = 8 - d \) and hence

\[
\alpha(d) = \frac{(4 - d)}{2} (-)^d.
\]

(8.1)

First of all, then, our generic Lagrangian (2.1) correctly describes the bosonic sector of the three-form field strength version of \( N = 1, D = 10 \) supergravity [38,39], where

\[
I_{10}(2) = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} \left( R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2\cdot3!} e^{-\phi} F_3^2 \right)
\]

(8.2)

since \( \alpha(2) = 1 \). The elementary solution of section 2 is therefore given by

\[
A = \frac{3}{8}(C - C_0),
\]

\[
B = -\frac{1}{8}(C - C_0),
\]

\[
\phi = \frac{1}{2}(C - C_0) + 2C_0,
\]

(8.3)

\[
20
\]
where $C_o = \phi_o/2$,
\[
e^{-C} = e^{-C_o} + \frac{k_2}{y^6},
\]
and
\[
k_2 = \kappa^2 T_2/3\Omega_7.
\]
This is the $D = 10$ string solution of Dabholkar et al [10], generalized to non-vanishing $\phi_o$ [14].

The generic Lagrangian (2.1) also describes the bosonic sector of the seven-form field strength version of $N = 1$, $D = 10$ supergravity [40], which is dual to (8.2), namely
\[
I_{10}(6) = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} \left( R - \frac{1}{2} (\partial \phi)^2 - \frac{e^\phi}{2\cdot7!} F_7^2 \right),
\]
since $\alpha(6) = -1$. The elementary solution of section 2 is therefore
\[
A = \frac{1}{8}(C - C_o), \quad B = -\frac{3}{8}(C - C_o),
\]
\[
\phi = -\frac{1}{2}(C - C_o) - 2C_o,
\]
where $C_o = -\phi_o/2$,
\[
e^{-C} = e^{-C_o} + \frac{k_6}{y^2},
\]
and
\[
k_6 = \kappa^2 T_6/\Omega_3,
\]
This is the $D = 10$ fivebrane solution of Duff and Lu [14]. The above solutions are dual with the elementary solutions of $I_{10}(2)$ corresponding to the solitonic solution of $I_{10}(6)$, and vice versa. From (6.14) and (6.21), the tensions obey
\[
\kappa^2 T_2 T_6 = |n| \pi,
\]
and the loop coupling constants obey
\[
g_2^2 g_6^8 = 1,
\]
in agreement with [13]. The mass per unit length of the string is given by
\[ M_2 = T_2 e^{\phi_s/2}, \] (8.12)
and the mass per unit five-volume of the fivebrane by
\[ M_6 = T_6 e^{-\phi_s/2}, \] (8.13)
As expected, the string gets heavier for weak fivebrane coupling and the fivebrane gets heavier for weak string coupling.

As shown in [10] and [14], both these solutions preserve half the spacetime supersymmetry.

Let us now turn our attention to \( D = 10 \) Type IIA supergravity whose action is given, in canonical variables, by
\[
I_{10}(IIA) = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} \left[ R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2.3!} e^{-\phi} F_3^2 - \frac{e^{3\phi/2}}{2.2!} F_2^2 - \frac{1}{2.4!} e^{\phi/2} F_4^4 - \frac{1}{8\kappa^2} \int F_4 \wedge F_4 \wedge A_2, \right] \tag{8.14}
\]
where \( F_4 = dA_3 + A_1 \wedge F_3 \) comparison with (8.1) shows that the kinetic terms for gravity, dilaton and antisymmetric tensors are correctly described by the generic action \( I_{10}(d) \) with \( d = 1, 2, 3 \) (i.e \( d = 7, 6, 5 \)). Both the elementary and solitonic \( N = 1 \) string and fivebrane solutions described above continue to provide solutions to Type IIA supergravity, as may be seen by setting \( F_2 = F_4 = 0 \). Again each preserve half the spacetime supersymmetry. (This establishes the existence of Type IIA superfinebras in \( D = 10 \), and these we surmise to be dual, in the sense of section 5, to Type IIA superstrings.) This observation is not as obvious as it may seem in the case of elementary fivebranes or solitonic strings, however, since it assumes that one may dualize \( F_3 \). Now the Type IIA action follows by dimensional reduction from the action of \( D = 11 \) supergravity, discussed in the next section. There exists no dual formulation of this action [41], in which \( F_4 \) is replaced by \( F_7 \), essentially because \( A_3 \) appears explicitly in the Chern-Simons term \( \int F_4 \wedge F_4 \wedge A_3 \). Since the \( F_4 \) and \( F_3 \) in \( D = 10 \) originate from \( F_4 \) in \( D = 11 \), this means that one cannot simultaneously dualize \( F_3 \) and \( F_4 \) but one may do either separately.\(^\dagger\) By partial integration

\(^\dagger\) We are grateful to H. Nishino for this observation.
one may choose to have no explicit \( A_3 \) dependence in the Chern-Simons term of (8.14) or no explicit \( A_2 \) dependence, but not both.

By setting \( F_2 = F_3 = 0 \), we find elementary membrane \((d = 3)\) and solitonic fourbrane \((\bar{d} = 5)\) solutions, and then by dualizing \( F_4 \), elementary fourbrane \((d = 5)\) and solitonic membrane \((\bar{d} = 3)\) solutions. From (3.15–18) and (8.1), the elementary membrane solution is given by

\[
A = \frac{5}{16}(C - C_o),
\]
\[
B = -\frac{3}{16}(C - C_o),
\]
\[
\phi = -\frac{1}{4}(C - C_o) - 4C_o,
\]
where \( C_o = -\phi_o/4 \),

\[
e^{-C} = e^{-C_o} + \frac{k_3}{y^3},
\]

and

\[
k_3 = 2\kappa^2 T_3/5\Omega_6.
\]

The fourbrane solution is given explicitly by

\[
A = \frac{3}{16}(C - C_o),
\]
\[
B = -\frac{5}{16}(C - C_o),
\]
\[
\phi = \frac{1}{4}(C - C_o) + 4C_o,
\]
where \( C_o = \phi_o/4 \),

\[
e^{-C} = e^{-C_o} + \frac{k_5}{y^3},
\]

and

\[
k_5 = 2\kappa^2 T_5/3\Omega_4,
\]

From (6.14) and (6.21), the tensions obey

\[
\kappa^2 T_3 T_5 = |n|\pi,
\]

and the loop coupling constants obey

\[
g_3^4 g_3^2 = 1,
\]

23
The mass per unit area of the membrane is given by

$$\mathcal{M}_3 = T_3 e^{-\phi_3/4},$$  \hspace{1cm} (8.23)

and the mass per unit four-volume of the fourbrane by

$$\mathcal{M}_5 = T_5 e^{\phi_4/4},$$  \hspace{1cm} (8.24)

Once again, these membrane and fourbrane solutions break one half of the spacetime supersymmetries [2] and hence there exist Type IIA supermembranes and Type IIA superfourbranes in accordance with a conjecture of Horowitz and Strominger [3], which we again expect to be dual to one another.

By setting $F_3 = F_4 = 0$, we find elementary particle ($d = 1$) and solitonic sixbrane ($\tilde{d} = 7$) solutions, and then by dualizing $F_2$, elementary sixbrane ($d = 7$) and solitonic particle ($\tilde{d} = 1$) solutions. From (3.15–18) and (8.1), the particle solution is given explicitly by

$$A = \frac{7}{16}(C - C_o),$$  

$$B = -\frac{1}{16}(C - C_o),$$  \hspace{1cm} (8.25)

$$\phi = -\frac{3}{4}(C - C_o) - \frac{4}{3}C_o,$$

where $C_o = -3\phi_o/4$,

$$e^{-C} = e^{-C_o} + \frac{k_1}{y^i},$$  \hspace{1cm} (8.26)

and

$$k_1 = 2\kappa^2 T_1/7\Omega_8,$$  \hspace{1cm} (8.27)

The sixbrane solution is given by

$$A = \frac{1}{16}(C - C_o),$$  

$$B = -\frac{7}{16}(C - C_o),$$  \hspace{1cm} (8.28)

$$\phi = \frac{3}{4}(C - C_o) + \frac{4}{3}C_o,$$

where $C_o = 3\phi_o/4$,

$$e^{-C} = e^{-C_o} + \frac{k_7}{y},$$  \hspace{1cm} (8.29)
and

\[ k_7 = 2\kappa^2 T_7 / \Omega_2, \]  

(8.30)

From (6.14) and (6.21), the tensions obey

\[ \kappa^2 T_1 T_7 = |n| \pi, \]  

(8.31)

and the loop coupling constants obey

\[ g_1 g_7^7 = 1, \]  

(8.32)

The mass of the particle is given by

\[ M_1 = T_1 \epsilon^{-3\phi_* / 4}, \]  

(8.33)

and the mass per unit six-volume of the sixbrane by

\[ M_7 = T_7 \epsilon^{3\phi_* / 4}, \]  

(8.34)

These particle and sixbrane solutions break one half of the spacetime supersymmetries and hence there exist Type II A superparticle and Type II A supersixbrane in accordance with the conjecture of Horowitz and Strominger [3], which we once more expect to be dual to one another.

Let us now turn our attention to Type IIB supergravity in \( D = 10 \) whose bosonic sector consists of the graviton \( g_{MN} \), a complex scalar \( \phi \), a complex 2-form \( A_2 \) (i.e \( d = 2 \) or, by duality, \( d = 6 \)) and a real 4-form \( A_4 \) (i.e. \( d = 4 \) which in \( D = 10 \) is self-dual!). Because of this self-duality of the 5-form field-strength \( F_5 \), there exists no covariant action principle of the kind (2.1) and, strictly speaking, our previous analysis ceases to apply. Nevertheless, we can apply the same logic to the equations of motion [42] and we find that the solutions again fall into the generic category given by (3.15–18). First of all, by truncation it is easy to see that the same string and fivebrane of \( N = 1 \) supergravity continue to solve the field equations of Type IIB. Moreover, they continue to break half the supersymmetries (but there are now twice as many since we start with \( N = 2 \) in \( D = 10 \) rather than \( N = 1 \)). Hence there exists Type IIB superstrings and superfivebranes, which are presumably dual.
On the other hand, if we set to zero the three form $F_3$ and make the ansatz (3.1–7) for the graviton, dilaton and $A_4$, setting to zero all other independent components of $F_5$ i.e those not related by the self-duality condition

$$F_5 = -^* F_5,$$

then we find the special case of (3.1–7) given by $d = \tilde{d} = 4$ and hence $\alpha = 0$ with $\phi = \text{constant}$. Explicitly, this self-dual threebrane is given by

$$A = \frac{1}{4} C,$$

$$B = -\frac{C}{4},$$

$$\phi = \text{constant},$$

with $C_\alpha = 0$. $C$ is given by

$$e^{-C} = 1 + \frac{k_4}{y^4}$$

and

$$k_4 = \kappa^2 T_4/2\Omega_5,$$

The mass per unit three-volume of the threebrane is

$$\mathcal{M}_4 = T_4,$$

Once again, we find that this solution preserves half the spacetime supersymmetries [22] and this establishes the existence of the self-dual Type IIB superthreebrane.

In summary, for $D = 10$ we have found elementary and solitonic string/fivebrane solutions for $N = 1$, Type IIA and Type IIB; particle/sixbrane solutions, membrane/fourbrane solutions for Type IIA only and self-dual threebrane solutions for Type IIB only, all of which are supersymmetric.

9. $D = 11$

We now turn our attention to $N = 1$, $D = 11$ supergravity. Before doing so, however, it is convenient to make the replacement (3.15) in (6.17) so that

$$I_D(d) = \frac{1}{2\kappa^2} \int d^Dx \sqrt{-g} e^{-(D-2)\alpha^2 C/4d} \left[ R - \frac{\alpha^2}{8} \left( 1 - \frac{\alpha^2(D-1)(D-2)}{2d^2} \right) (\partial C)^2 - \frac{1}{2 \cdot (d+1)!} F_{d+1}^2 \right],$$

26
where we have set $C_0 = 0$ for simplicity. If we now focus on the case $(D = 11, \, d = 3)$ we find from (3.18) that
\[\alpha(3) = 0,\] (9.2)
and hence
\[I_{11}(3) = \frac{1}{2\kappa^2} \int d^{11}x \sqrt{-g} \left[ R - \frac{1}{2.4!} F_4^2 \right].\] (9.3)
This is to be compared with the bosonic sector of $D = 11$ supergravity
\[I(D = 11SUGRA) = \frac{1}{2\kappa^2} \int d^{11}x \sqrt{-g} \left[ R - \frac{1}{2.4!} F_4^2 \right] - \int F_4 \wedge F_4 \wedge A_3.\] (9.4)
As discussed in section 7, there is no dualized form of this action since $A_3$ enters explicitly. We can however find elementary membrane solution. Once again, this is just a special case of our general solutions (3.15–18). For $d = 3, \, \d = 6, \, \alpha(3) = 0$ we find explicitly
\[A = \frac{1}{3} C,\] \[B = \frac{1}{6} C,\] (9.5)
$C$ is given by
\[e^{-C} = 1 + \frac{k_3}{y^6},\] (9.6)
and
\[k_3 = \kappa^2 T_3/3\Omega_7.\] (9.7)
The mass per unit area of the membrane is
\[M_3 = T_3.\] (9.8)
This is Duff-Stelle [12] solution which breaks half the supersymmetries and corresponds to the eleven-dimensional supermembrane of Bergshoeff, Sezgin and Townsend [43].

10. **Double dimensional reduction and $D < 10$ supersymmetry.**

Simple dimensional reduction allows us to derive the actions $I_D(d)$ and $S_d$ for a $(d-1)$-brane moving in a $D$-dimensional spacetime from the actions $I_{D+1}(d)$ and $S_d$ corresponding
to a \((d - 1)\)-brane in a \((D + 1)\)-dimensional spacetime. This corresponds to

\[
D + 1 \rightarrow D, \\
d \rightarrow d, \\
\hat{d} + 1 \rightarrow \hat{d},
\]

and takes us vertically on the brane-scan. Double dimensional reduction [29], on the other hand, allows us to derive the actions \(I_D(d)\) and \(S_d\) for a \((d - 1)\)-brane moving in \(D\)-dimensional spacetime from the actions \(I_{D+1}(d+1)\) and \(S_{d+1}\). This corresponds to

\[
D + 1 \rightarrow D, \\
d + 1 \rightarrow d, \\
\hat{d} \rightarrow \hat{d},
\]

and takes us diagonally on the brane-scan. The first example of this was to rederive the Type IIA superstring in \(D = 10\) from the supermembrane in \(D = 11\) [29]. This process thus allows us, for example, to rederive the Dabholkar et al superstring [10] solution in \(D = 10\) from the Duff-Stelle supermembrane [12] solution in \(D = 11\).

To see how it works in general, let us denote all \((D + 1, d + 1)\)-dimensional quantities by a hat and all \((D, d)\) dimensional quantities without. Then with

\[
\hat{X}^M = (X^M, X^d), \quad M = 0, 1, \ldots, (d - 1), (d + 1), \ldots, D - 1 \\
\hat{\xi}^\mu = (\xi^i, \xi^d),
\]

double dimensional reduction consists in setting

\[
\xi^d = X^d,
\]

taking \(X^d\) to be the coordinate on a circle of radius \(R\), and discarding all but the zero modes. In practice, this means taking the background fields \(\hat{\phi}, \hat{g}_{MN}\) and \(\hat{A}_{MN\ldots M_d}\) to be independent of \(X^d\). To recover \(S_d\), with only background fields \(\phi, g_{MN}\) and \(A_{M_1 M_2 \ldots M_{d-1}}\), a further truncation is necessary. Specifically we write

\[
\hat{g}_{MN}(\sigma - \text{model}) = e^{-2\beta \phi/d+1} \begin{pmatrix}
  g_{MN}(\sigma - \text{model}) & 0 \\
  0 & e^{2\beta \phi}
\end{pmatrix},
\]

\(28\)
where \( \beta \) is a, for the moment, arbitrary constant and

\[
\hat{A}_{012\ldots d+1} = A_{012\ldots d},
\]  

(10.6)

with other components set to zero. The condition (10.5) ensures from (2.18) that

\[
\sqrt{-\gamma} = \sqrt{-\hat{\gamma}},
\]  

(10.7)

and hence, together with condition (10.6), we recover the correct \( \sigma \)-model action for \( S_{d-1} \) starting from \( \hat{S}_d \) provided

\[
2\pi R\hat{T}_{d+1} = T_d.
\]  

(10.8)

We fix \( \beta \) and the relation between \( \hat{\phi} \) and \( \phi \) by requiring that we obtain the correct background field action \( I_D(d) \) starting from \( I_{D+1}(d+1) \). So from (6.17)

\[
e^{-(D-1)\hat{\alpha}/2(d+1)}\sqrt{-\hat{\gamma}} \left[ \hat{R} - \frac{1}{2} \left( 1 - \frac{\hat{\alpha}^2(D(D-1))}{2(d+1)^2} \right) (\partial \hat{\phi})^2 \right]
\]

\[
= e^{-(D-2)\alpha/2d}\sqrt{-\gamma} \left[ R - \frac{1}{2} \left( 1 - \frac{\alpha^2(D-1)(D-2)}{2d^2} \right) (\partial \phi)^2 \right],
\]  

(10.9)

which gives

\[
\hat{\phi} = \delta \phi, \\
\frac{(D-1)\hat{\alpha}}{2(d+1)} \delta = \frac{(D-2)\alpha}{2d} - \frac{\hat{\alpha}}{d+1},
\]

\[
1 - \frac{\alpha^2(D-1)(D-2)}{2d^2} = \delta^2 \left( 1 - \frac{\alpha^2 D(D-1)}{2(d+1)^2} \right) - 4\beta \frac{d+1}{d+1} \frac{(D-2)\alpha}{2d}
\]

\[
+ 2\beta^2 \frac{D(D-1) - 2(d+1)(d+1)}{(d+1)^2},
\]  

(10.10)

and hence

\[
\beta = \frac{2}{d \alpha},
\]  

(10.11)

\[
\delta = \frac{\hat{\alpha}}{\alpha},
\]  

(10.12)

from solving eqs. (10.10). We also require

\[
\hat{\kappa}^2 = 2\pi R\kappa^2.
\]  

(10.13)
Note that the Dirac quantization rule (6.14) involving $\kappa^2$ and $T$ follows from that involving $\tilde{\kappa}^2$ and $\tilde{T}$ on using (10.8) and (10.13). In canonical variables, we have

$$\hat{g}_{MN}(\text{canonical}) = e^{-2\hat{d}\phi/\alpha(d)(d+\hat{d})(d+1+\hat{d})}g_{MN}(\text{canonical}),$$
$$\hat{g}_{dd}(\text{canonical}) = e^{2\hat{d}\phi/(d+1+\hat{d})\alpha(d)}.$$  

(10.14)

As an application of simultaneous dimensional reduction, we may derive the Dabholkar et al elementary string solution (8.3–5) in $D = 10$ from the Duff-Stelle membrane solution in $D = 11$. The $D = 10$ fields $g_{MN}, A_{MN}$ and $\phi$ are given by

$$\hat{g}_{MN} = e^{-\phi/6}g_{MN}(\text{canonical}),$$
$$\hat{g}_{22} = e^{4\phi/3},$$
$$\hat{A}_{012} = A_{01}.$$  

(10.15)

[Curiously, the metric $\hat{g}_{MN}$ in (10.15) bears the same relation to $g_{MN}$ (canonical) as does the fivebrane $\sigma$-model metric in (6.16) since $\alpha(d = 2) = 1$ and $\hat{d} = 6$. This phenomenon happens in general whenever $\hat{\alpha} = 0$ i.e. for $(d + 1 = 3, \hat{d} = 6), (d + 1 = 4, \hat{d} = 4)$ and $(d + 1 = 6, \hat{d} = 3)]$. Similarly starting from sixbrane in $D = 10$ we may proceed diagonally down the brane-scan to a particle in $d = 4$. It is not difficult to show that the solutions so obtained will continue to preserve exactly one half of the supersymmetries. Starting from the $d \leq 7$ solutions in $D = 10$ we can thus fill out the triangle of supersymmetric extended objects described in [2].

11. $D = 6$: The self-dual string

If we proceed by double dimensional reduction from the super-fivebrane in $D = 10$ we arrive at a super $(\hat{d} = 2)$ string in $D = 6$ for which $\alpha(\hat{d} = 2) = -\alpha(d = 2)$ i.e which is dual to the elementary super string and related by a strong/weak coupling replacement $\phi \rightarrow -\phi$. Compare (6.23) with (6.24).

However, there is another supersymmetric solitonic string in $D = 6$: the self-dual superstring which falls outside our previous discussions and requires a special treatment. This is the $D = 6$ counterpart of the self-dual superthreebrane in $D = 10$. Our starting point is the $N = 2, D = 6$ self-dual supergravity [44,45] which, in common with the Type
IIB superstring in $D = 10$, admits covariant field equations, but no manifestly covariant field equations. It describes a graviton $\epsilon^A_M$, two left-handed gravitini $\psi_{Ma}$ and one tensor field $B_{MN}$ with self-dual field strength $G_{MNP}$. The gravitini transformation rules are (in our notation)

$$\delta \psi_M = \nabla_M \varepsilon - \frac{1}{8} G_{MNP} \Gamma^{NP} \varepsilon.$$  \hfill (11.1)

So if we make a two/four split as in section 3 with

$$\Gamma_A = (\gamma_\alpha \otimes 1, \gamma^3 \otimes \Sigma_m), \quad \Gamma^7 = \gamma^3 \otimes \Gamma^5,$$

$$\gamma^3 = \gamma^0 \gamma^1, \quad \Gamma^5 = \Sigma^2 \Sigma^3 \Sigma^4 \Sigma^5,$$  \hfill (11.2)

the criterion for unbroken supersymmetry, $\delta \psi_M = 0$, reduces to

$$\partial_\mu \varepsilon - \frac{1}{2} \gamma^3 \gamma_\mu \otimes \Sigma^n (\partial_n A + \frac{1}{2} \varepsilon^{-2A} \partial_n \epsilon^C \gamma^3 \varepsilon) = 0,$$

$$\partial_m \varepsilon + \frac{1}{2} \partial_m B \varepsilon - \frac{1}{2} (\delta^m_n + \Sigma^m_n) (\partial_n B - \frac{1}{2} \varepsilon^{-2A} \partial_m \epsilon^C \gamma^3 \varepsilon) = 0,$$  \hfill (11.3)

and hence supersymmetry requires

$$C = 2A, \quad B = -A, \quad \varepsilon = \epsilon^{-B/2} \varepsilon_\alpha,$$  \hfill (11.4)

where $\varepsilon_\alpha$ obeys $\gamma^3 \varepsilon_\alpha = -\varepsilon_\alpha$, and one half of the supersymmetries is broken.

The bosonic equations of motion are

$$R_{MN} - \frac{1}{2} g_{MN} R = \frac{1}{4} G^{PQ} G_{NPQ}$$  \hfill (11.5)

$$G_{MNP} = -G_{MNP},$$  \hfill (11.6)

and substituting (11.4) yields

$$\epsilon^{6A} \delta^{mn} \partial_m \partial_n \epsilon^{-2A} = 0,$$  \hfill (11.7)

for the $\mu \nu$ components of the Einstein equation and

$$\epsilon^{2A} \delta^{mn} \partial_m \partial_n \epsilon^{-2A} = 0,$$  \hfill (11.8)

for the $mn$ components. So

$$\epsilon^{-2A} = 1 + \frac{k_2}{y^2}.$$  \hfill (11.9)
All the properties of the dyonic self-dual threebrane [22] apply, mutatis mutandis, to the dyonic self-dual string, including Dirac quantization rules and the saturation of the Bogomol’nyi bound.

The effective bosonic equations of motion of this string are

\[ \partial_i (\sqrt{-g} \gamma^{ij} \partial_j X^N g_{MN}) - \frac{1}{2} \sqrt{-g} \gamma^{ij} \partial_i X^N \partial_j X^P \partial_M g_{NP} \]

\[ = \frac{1}{2} G_{MNP} \partial_i X^N \partial_j X^P \varepsilon^{ij} , \]

but, since \( G_{MNP} = -G_{MNP} \), there is no manifestly covariant world sheet action. It would be interesting to include the fermionic degrees of freedom and construct the spacetime supersymmetric, \( \kappa \)-symmetric, Green-Schwarz string equations, but this has not yet been done.

12. \( D = 4 \): Electric-Magnetic Duality

Alternatively, we may proceed vertically down the brane-scan as far as \( \ddot{d} = 1 \). Thus starting with a particle in \( D = 10 \) we arrive at a particle in \( D = 4 \). However, this solution will have an \( \alpha \) parameter \( \alpha(d = 1) = +\sqrt{3} \) opposite in sign to the \( \alpha(\ddot{d} = 1) = -\sqrt{3} \) solution obtained by double dimensional reduction from the sixbrane in \( D = 10 \). The two Lagrangians are given by

\[ I_4(1) = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} \left( R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{4} e^{-\sqrt{3} \phi} F_{\mu\nu} F^{\mu\nu} \right), \]  

\[ \tilde{I}_4(1) = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} \left( R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{4} e^{+\sqrt{3} \phi} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} \right), \]

where

\[ \tilde{F}_{\mu\nu} = e^{-\sqrt{3} \phi} \ast F_{\mu\nu} . \]

If, for simplicity, we set \( \phi_0 = 0 \), then the action \( I_4(1) \) admits the elementary solution

\[ ds^2 = - \left( 1 + \frac{k_1}{y} \right)^{-1/2} dt^2 + \left( 1 + \frac{k_1}{y} \right)^{1/2} (dy^2 + y^2 d\theta^2 + y^2 \sin^2 \theta d\phi^2) , \]

\[ e^{2\phi} = \left( 1 + \frac{k_1}{y} \right)^{-\sqrt{3}} , \]

\[ B_\phi = - \left( 1 + \frac{k_1}{y} \right)^{-1} , \]
where
\[ k_1 = \frac{\kappa^2 T_1}{2\pi}, \]  
(12.5)
corresponding to an electric monopole with mass
\[ m = T_1, \]  
(12.6)
and electric charge
\[ e = \sqrt{2\kappa m}. \]  
(12.7)

\( I_4(1) \) also admits the solitonic solution
\[ ds^2 = -\left(1 + \frac{k_1}{y}\right)^{-1/2} dt^2 + \left(1 + \frac{k_1}{y}\right)^{1/2} \left(dy^2 + y^2 d\theta^2 + y^2 \sin^2 \theta d\phi^2\right), \]
\[ e^{2\phi} = \left(1 + \frac{k_1}{y}\right)^{\sqrt{2}}, \]  
(12.8)
\[ \frac{1}{\sqrt{2\kappa}} F_{\theta\phi} = \frac{g}{4\pi}, \]
where
\[ k_1 = \frac{\kappa^2 T_1}{2\pi} = \frac{n}{2T_1}, \]  
(12.9)
corresponding to a magnetic monopole with mass \( T_1 \) and magnetic charge obeying
\[ eg = 2\pi n. \]  
(12.10)

For the dual action \( \tilde{I}_4(1) \), the electric and magnetic solutions are interchanged.

These solutions are precisely the extreme mass = charge limits of the black-hole solutions of \( D = 4, \ N = 8 \) supergravity discussed by Gibbons and Perry [30]. See also Han et al [46] who obtained the same solution from \( D = 11 \) supergravity. Gibbons and Perry pointed out that, considered as solutions of \( N = 8 \) supergravity in \( D = 5 \), the monopole solitons fit into the same supermultiplets as the elementary electric monopoles, and went on to speculate that there exists a dual theory for which the roles of elementary and solitonic particles are interchanged. In the light of the results of the present paper, we may re-interpret this electric-magnetic duality conjecture in \( D = 4 \) as a particle/sixbrane duality conjecture in \( D = 10 \). (Note that the values of \( \alpha \) considered here i.e. \( \pm \sqrt{3} \) differ from the \( \alpha = \pm 1 \) considered by Shapere et al [47] and Kallosh et al [48] in the same context of electric-magnetic duality and to which, therefore, the above remarks do not apply.)

33
13. $D = 4$ axionic instanton

Another special case of interest corresponds to $\tilde{d} = 0$, $D = 4$, $\alpha(2) = 2$ (solitonic solution). Since there is now no time coordinate, this corresponds to a Euclidean instanton. From (5.3), we have

$$ds^2 = \delta_{\mu\nu}dy^\mu dy^\nu, \quad m = 1, 2, 3, 4$$

$$e^\phi = 1 + \frac{k_0}{y^2}. \quad (13.1)$$

$$\frac{1}{\sqrt{2}\kappa} F_3 = g_0 \bar{\epsilon}_3 / \Omega_3$$

The metric is flat, and the energy-momentum tensor vanishes. This solution is just the axionic instanton first discovered by Soo-Jong Rey [31].

14. Black branes

In the case $D = 10$, Horowitz and Strominger discovered two parameter solitonic solutions of the theory (2.1) in the cases $1 \leq d \leq 7$ which displayed event horizons: the “black p-branes” [3]. The two parameters are the mass per unit volume $M_\tilde{d}$ and the charge per unit volume $g_\tilde{d}$, which satisfy the Bogolmol’nyi bound $\kappa M_\tilde{d} \geq g_\tilde{d} / \sqrt{2}$. In this section, we generalize their results to arbitrary $D$. We also show that in the cases $1 \leq d \leq 7$ and $1 \leq \tilde{d} \leq 7$, the extreme black p-branes i.e those obeying the mass = charge limit $\kappa M_\tilde{d} = g_\tilde{d} / \sqrt{2}$, coincide with the Type II super p-branes [2].
Using the canonical metric, the \((\tilde{d} - 1)\) brane black soliton solution may be written for all \(\tilde{d} \geq 1\) as

\[
\begin{align*}
    ds^2 &= - \left[ 1 - \left( \frac{r_+}{r} \right)^d \right] \left[ 1 - \left( \frac{r_-}{r} \right)^d \right]^{\frac{\tilde{d}}{(d+\tilde{d})}} dt^2 \\
    &+ \left[ 1 - \left( \frac{r_+}{r} \right)^d \right]^{-1} \left[ 1 - \left( \frac{r_-}{r} \right)^d \right]^{\frac{2}{d+1}} dr^2 \\
    &+ r^2 \left[ 1 - \left( \frac{r_-}{r} \right)^d \right]^{\frac{2}{d+1}} d\Omega_{d+1}^2 \\
    &+ \left[ 1 - \left( \frac{r_-}{r} \right)^d \right]^{\alpha(d)} dx^i dx_i, \quad i = 1 \ldots \tilde{d} - 1, \\
    e^{-2\phi} &= \left[ 1 - \left( \frac{r_-}{r} \right)^d \right]^{\alpha(d)}, \\
    \frac{1}{\sqrt{2\kappa}} F_{d+1} &= g_\tilde{d} \xi_{d+1}/\Omega_{d+1},
\end{align*}
\]

(14.1)

where the magnetic charge \(g_\tilde{d}\) and the mass per unit \((\tilde{d} - 1)\)-volume \(\mathcal{M}_{\tilde{d}}\) are related to \(r_\pm\) by [49]

\[
    g_\tilde{d} = \frac{\Omega_{d+1}}{\sqrt{2\kappa}} d(r_+ r_-)^{d/2}, \quad (14.2)
\]

\[
    \mathcal{M}_{\tilde{d}} = \frac{\Omega_{d+1}}{2\kappa^2} [(d + 1)r_+^d - r_-^d]. \quad (14.3)
\]

We note that there are consistent with the Bogolmol’nyi bound (5.4) with \(\phi_o = 0\). The solutions poses an \(R \times SO(d+2) \times E(\tilde{d} - 1)\) symmetry where \(E(n)\) denotes the \(n\)-dimensional Euclidean group. The solutions exhibit an event horizon at \(r = r_+\) and an inner horizon at \(r = r_-\). In the special case \(D = 11\), \(\tilde{d} = 3, 6\) they reduce to the black membrane and black fivebrane of Guven [6]. In the special case \(D = 10\) i.e \(\tilde{d} = 8 - d\), they reduce to the Horowitz-Strominger black \(p\)-brane solutions [3]. In the special case \(D = 4\), \(\tilde{d} = 1\) they reduce to the dilaton black hole solution of Gibbons and Perry [30]. Two special cases of interest are the zero charge solutions \((r_- = 0)\) and the extreme mass = charge solutions \((r_+ = r_-)\). In the first case the dilaton and antisymmetric tensor are trivial and the metric reduces to

\[
\begin{align*}
    ds^2 &= -V dt^2 + V^{-1} dr^2 + r^2 d\Omega_{d+1}^2 + dx^i dx_i, \\
    V &= 1 - \left( \frac{r_+}{r} \right)^d.
\end{align*}
\]

(14.4)
Gregory and Laflamme have argued that (in the \( D = 10 \) case) these solutions are classically unstable \([7]\). In the second case, remarkably enough, at the external limit \( r_+ = r_- \) the metric component \( g_{oo} \) becomes equal to the one multiplying \( dx^i dx^i \) and the symmetry is enlarged to \( SO(d + 2) \times P(\tilde{d}) \):

\[
ds^2 = \left[ 1 - \left( \frac{r_-}{r} \right)^d \right]^{d/(d + \tilde{d})} dx^\mu dx_\mu \]

\[
+ \left[ 1 - \left( \frac{r_-}{r} \right)^d \right]^{\frac{\tilde{d}}{d}} \left\{ \left[ 1 - \left( \frac{r_-}{r} \right)^d \right]^{-2} dr^2 + r^2 d\Omega^2_{d+1} \right\},
\]

\[
e^{-2\phi} = \left[ 1 - \left( \frac{r_-}{r} \right)^d \right]^{\alpha(d)},
\]

\[
\frac{1}{\sqrt{2\kappa}} F_{d+1} = g_{d \tilde{d}} \hat{\Omega}_{d+1}/\Omega_{d+1}.
\]

It is convenient to introduce the change of variables \( y^d = r^d - r_-^d \), then (14.5) becomes

\[
ds^2 = \left[ 1 + \left( \frac{r_-}{y} \right)^d \right]^{-d/(d + \tilde{d})} dx^\mu dx_\mu + \left[ 1 + \left( \frac{r_-}{y} \right)^d \right]^{\tilde{d}/(d + \tilde{d})} (dy^2 + y^2 d\Omega^2_{\tilde{d}+1}),
\]

\[
e^{-2\phi} = \left[ 1 + \left( \frac{r_-}{y} \right)^d \right]^{-\alpha(d)},
\]

\[
\frac{1}{\sqrt{2\kappa}} F_{d+1} = g_{d \tilde{d}} \hat{\Omega}_{d+1}/\Omega_{d+1}.
\]

But in the case \( 1 \leq d \leq 7, 1 \leq \tilde{d} \leq 7 \), these are precisely the super \( p \)-branes, so \( r_+ = r_- \) also corresponds to the appearance of supersymmetry.

It is also possible to find *elementary* black \((d - 1)\)-branes with parameters \( M_d \) and \( \epsilon_d \) obeying the bound \( \kappa M_d \geq \sqrt{2} \epsilon_d \), by including a source term on the right hand side of the equations. In this case however, it would be necessary to relax the equality of the kinetic and WZW term coefficients in (2.3) to allow for mass \( \neq \) charge. (This equality is forced on us in the supersymmetric case, by virtue of \( \kappa \)-symmetry [43]).
15. Black and super p-branes: Singular or non-singular?

In this section, we would like to classify the singular nature of black and super p-branes discussed in previous sections thus generalizing the results of [33]. The black \((d - 1)\)-brane solutions can be obtained from the black \((\tilde{d} - 1)\)-brane solutions simply by sending \(d \leftrightarrow \tilde{d}\) in previous section. The Ricci scalar of black \((d - 1)\)-brane calculated in terms of \((n - 1)\)-brane variables is

\[
R = \left[1 - \left(\frac{r_-}{r}\right)^{\tilde{d}}\right]^{-\frac{\alpha^2(d)}{2d} + \frac{\alpha(n)\alpha(d)}{2n}} \times \frac{Q^2}{r^{2(d+1)}} \times \left\{\frac{1}{8} \left[2(d + 2)(1 - d) - \frac{(D - 1)(D - 2)}{2} \left(\frac{\alpha^2(d)}{n^2} - \frac{4\tilde{d}^2}{(D - 2)^2}\right)\right] \right. \\
\left. \left(\frac{r_-}{r}\right)^{\tilde{d} - 1} \right\} \left(\frac{1}{8} \left[2(d + 2)(1 - d) - \frac{(D - 1)(D - 2)}{2} \left(\frac{\alpha^2(d)}{n^2} + \frac{2\tilde{d}}{D - 2}\right)\right] \right). \tag{15.1}
\]

We note that for either \(n = d\) or \(n = \tilde{d}\), i.e. written in its own variables or its dual ones, the \(R\) in (15.1) always blows up as \(r\) goes to \(r_-\). The reason is that \(S \equiv -\frac{\alpha^2(d)}{2d} + \frac{\alpha(n)\alpha(d)}{2n}\) is now less than one. For \(n = d\), \(S = -\frac{1}{2}(\frac{1}{d} + \frac{1}{\tilde{d}})\alpha^2(d) < 0\), and for \(n = \tilde{d}\), \(S = 0\) since \(\alpha(\tilde{d}) = -\alpha(d)\). In conclusion, physically interesting black p-branes always display singularities at \(r = r_-\).

The situation for super p-branes is quite different. The calculations proceed along the same lines as [33]. Here we omit the details and simply state the results. We can always choose suitable variables to get rid of the singularities (for example, the dual variables) or else there is no singularity at all (as for example in the self-dual threebrane). We can also calculate the proper time for a \((n - 1)\)-brane falling into a \((d - 1)\)-brane. We find that only for strings and their dual objects, the corresponding proper time is infinite, which agrees with what we have discussed in [33]. For any other object and its dual, the corresponding proper time is always finite although there is no curvature singularity when written in terms of the dual variables, which contradicts our naive expectations. For the self-dual cases, the self-dual threebrane [22] is free of singularity and the corresponding proper time is finite, and similarly for the self-dual string. Any extended object, except
for self-dual string and threebrane, has a curvature singularity when written in terms of its own variables and the corresponding proper time is finite. From the above, it is easy to see that only strings (except for the self-dual one) satisfy our naive expectations.

Appendix A. Comparison with Brans-Dicke Theory

The action for Brans-Dicke gravity (generalized from 4 to D dimensions) may be written in terms of a scalar field \( \eta \) and some metric \( g_{MN} \) (BD)

\[
I(\text{Brans – Dicke}) = \frac{1}{2\kappa^2} \int d^Dx \sqrt{-g} \left[ \eta R - \frac{\omega}{\eta} (\partial \eta)^2 \right] + \int d^Dx \mathcal{L} (\text{matter, } g),
\]

(A.1)

where \( \omega \) is a free parameter and where, by construction, \( \mathcal{L} \) (matter, \( g \)) is independent of \( \eta \). In comparing this to our general action \( I_D(d) \) we have to decide what is meant by \( \mathcal{L} \) (matter, \( g \)). Let us first suppose that this refers not to the antisymmetric tensor action of (2.1) but to the \((d - 1)\)-brane action \( S_d \) of (2.3). Then we must make the identification

\[
g_{MN}(BD) = g_{MN}(d),
\]

(A.2)

where \( g_{MN}(d) \) is the \((d - 1)\)-brane \( \sigma \)-model metric of (6.15). Comparison with (6.17) then yields the identifications

\[
\eta = e^{-(D-2)\alpha(d)\phi/2d},
\]

(A.3)

\[
\omega \frac{2d^2}{(D - 2)\alpha^2(d)} - \frac{D - 1}{D - 2} = -\frac{(D - 1)(d - 2) - d^2}{(D - 2)(d - 2) - d^2},
\]

(A.4)

where we have used (3.18). It is interesting to note, for example, that in \( D = 10 \) strings \((d = 2)\) correspond to \( \omega = -1 \), fivebranes \((d = 6)\) to \( \omega = 0 \) and threebranes \((d = 4)\) to \( \omega = \infty \). However, if we now include in \( \mathcal{L} \) (matter, \( g \)) the action for a \((d' - 1)\) antisymmetric tensor written in \((d - 1)\)-brane variables, namely

\[
\frac{1}{2\kappa^2} \int d^Dx \sqrt{-g} e^{-(D-2)\alpha(d)\phi/2d} e^{[d'\alpha(d')-d\alpha(d)]\phi/d} \frac{1}{2(d' + 1)!} F_{d' + 1}^2
\]

(A.5)

then our theory will no longer be of the Brans-Dicke form unless

\[
d' \alpha(d') - d\alpha(d) = \left( \frac{D - 2}{2} \right) \alpha(d)
\]

(A.6)
If $d = d'$ this will not be satisfied unless $(D - 2)\alpha(d)$ vanishes in which case $\omega = \infty$. If, on the other hand, we omit the action $S_d$ and take $\mathcal{L}$ (matter, $g$) to indicate the antisymmetric tensor action alone, then we may re-interpret (A.5) as the tree-level action for a $(d' - 1)$-brane written in $\sigma$-model variables. Then (A.6) is no longer a restriction on the dimension of the extended object but only on the variables we choose to write the action.

**Acknowledgments**

We would like to thank J. Dixon, G. Horowitz, R. Khuri, H. La, R. Minasian, J. Rahmfeld, E. Sezgin and A. Strominger for helpful discussions.
References