QCD Quark Condensate from SUSY and the Orientifold Large-$N$ Expansion

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Abstract

We estimate the quark condensate in one-flavor massless QCD from the known value of the gluino condensate in SUSY Yang-Mills theory using our newly proposed “orientifold” large-$N$ expansion. The numerical result for the quark condensate renormalized at the scale 2 GeV is then given as a function of $\alpha_s(2 \text{ GeV})$ and of possible corrections from sub-leading terms. Our value can be compared with the quark condensate in (quenched) lattice QCD or with the one extracted from the Gell-Mann–Oakes–Renner relation by virtue of non-lattice determinations of the quark masses. In both cases we find quite a remarkable agreement.
1 Introduction

In non-supersymmetric theories, such as QCD, it is very difficult, if possible at all, to perform reliable analytic calculations in the non-perturbative regime. In the supersymmetric version of the theory the situation is much better, due to holomorphy. In particular the exact gluino condensate [1] can be evaluated in \( \mathcal{N} = 1 \) Super-Yang-Mills.

In previous publications [2, 3] we suggested a precise way of copying non-perturbative results from a supersymmetric theory to a certain non-supersymmetric theory named orientifold field theory (due to its realization via orientifold type-0 string theory [4]). The orientifold theory is an SU(\( N \)) gauge theory coupled to a Dirac fermion in the two-index antisymmetric representation, \( \mathbb{1} + \mathbb{1} \). Similarly, one can consider [5] a generalized orientifold QCD (or QCD\textsubscript{OR}), which consists of \( N_f \) flavors of Dirac fermions in the antisymmetric representation (\( N_f \geq 1 \)).

Our purpose here is to carry out an explicit calculation of the quark condensate in one-flavor QCD anticipated in Ref. [3]. Let us briefly recall the idea behind such a calculation. Consider three one-parameter families of gauge theories, the above-mentioned parameter being \( N \), of their common gauge group SU(\( N \)):

- Pure Yang Mills (YM) theory also known as gluodynamics;
- QCD\textsubscript{F}, i.e. standard ’t Hooft’s extension of QCD at arbitrary \( N \) (the number of quarks in the fundamental plus anti-fundamental representation \( N_f \) is kept fixed);
- QCD\textsubscript{A}, i.e. the SU(\( N \)) gauge theory with \( N_f \) Majorana fermions in the adjoint representation.

In Ref. [3] we made a simple observation that, as \( N \) increases, the generalized orientifold theory QCD\textsubscript{OR} interpolates between the three other theories above. Indeed, at \( N = 2 \), the fermions of QCD\textsubscript{OR} are gauge singlets, and, therefore, QCD\textsubscript{OR} reduces to YM. At \( N = 3 \), QCD\textsubscript{OR} obviously coincides with QCD\textsubscript{F}. Moreover, at \( N \to \infty \) the bosonic sector of QCD\textsubscript{OR} goes into that of QCD\textsubscript{A} — a straightforward generalization [5] of the results of Ref. [2]. This last observation becomes particularly interesting for one massless flavor, since in this case the limiting large-\( N \) theory QCD\textsubscript{A} is nothing but the
supersymmetric generalization of YM theory — SYM theory also known as supersymmetric gluodynamics. We will limit our attention to this case hereafter.

A consistency check of the above statements can be made by comparing the coefficients of the $\beta$ functions of these theories at different values of $N$, as well as the anomalous dimensions of the corresponding fermion bilinears $\bar{\psi}\psi$. In Table 1 we present this check for the $N_f = 1$ case under discussion.

<table>
<thead>
<tr>
<th>Theory (Coefficients)</th>
<th>YM</th>
<th>QCD$_F$</th>
<th>QCD$_{OR}$</th>
<th>SYM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_0$</td>
<td>$\frac{11}{3} N$</td>
<td>$\frac{11}{3} N - \frac{2}{3}$</td>
<td>$3N + \frac{4}{3}$</td>
<td>$3N$</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>$\frac{17}{3} N^2$</td>
<td>$\frac{17}{3} N^2 - \frac{13}{6} N + \frac{1}{2N}$</td>
<td>$3N^2 + \frac{19}{3} N - \frac{4}{N}$</td>
<td>$3N^2$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$\ast$</td>
<td>$\frac{3(N^2-1)}{2N}$</td>
<td>$\frac{3(N-2)(N+1)}{N}$</td>
<td>$3N$</td>
</tr>
</tbody>
</table>

Table 1.

We use the standard definition of the coefficients of the $\beta$ function from PDG, see Ref. [6],

$$\mu \frac{\partial \alpha}{\partial \mu} \equiv 2\beta(\alpha) = -\frac{\beta_0}{2\pi} \alpha^2 - \frac{\beta_1}{4\pi^2} \alpha^3 + \ldots$$  \hspace{1cm} (1)

The coefficients can be found in [7], where formulae up to three loops are given. The anomalous dimension $\gamma$ of the fermion bilinear operators $\bar{\Psi}\Psi$ is normalized in such a way that

$$(\bar{\Psi}\Psi)_Q = \kappa^{\gamma/\beta_0} (\bar{\Psi}\Psi)_\mu \quad \kappa \equiv \frac{\alpha(\mu)}{\alpha(Q)}$$, \hspace{1cm} (2)

and $\mu$ and $Q$ denote the normalization points. For our present purposes we can limit ourselves to the two-loop $\beta$-functions and the one-loop anomalous dimensions. We can check easily that the various coefficients of QCD$_{OR}$ go smoothly from those of YM ($N = 2$) through those of QCD$_F$ ($N = 3$) to those of SYM theory at $N \to \infty$.

Since many non-perturbative properties of SYM theory are known, the large-$N$ expansion will provide us with information on the non-perturbative
behavior of one-flavor QCD at $N = 3$, modulo $1/N \sim 1/3$ corrections. The planar equivalence method is applicable to a large class of bosonic correlators and can be tested, in principle, in lattice calculations. In this letter we will concentrate our attention on a one-point function, the quark condensate, which has been explicitly computed in SYM theory, and measured on the lattice.

2 Renormalization-group-invariant quantities with a smooth large-$N$ limit

In order to carry out our calculation we need to define, for each theory, renormalization-group-invariant (RGI) fermion-bilinear operators and compare their vacuum expectation values (VEV’s) with the appropriate power of the corresponding fundamental RGI scale $\Lambda$. Furthermore, we would like always to deal with quantities that can be expanded at large $N$ and fixed 't Hooft ($N$-independent) coupling $\lambda$,

$$\lambda \equiv \frac{g^2 N}{8\pi^2} = \frac{\alpha N}{2\pi},$$

as a (possibly asymptotic) power series in $1/N$. It turns out that this latter requirement calls for a slightly unconventional definition of the above quantities since at fixed $\lambda$ a non-integer power of $g^2$ behaves as a non-integer power of $N$ and is, thus, non-analytic at $N = \infty$.

The standard (two-loop) definition of the scale parameter $\Lambda$, which follows from the conventions of Ref. [6], is

$$\Lambda_{\text{standard}} = \mu \left( \frac{16\pi^2}{\beta_0 g^2(\mu)} \right)^{\beta_1/\beta_0^2} \exp \left( -\frac{8\pi^2}{\beta_0 g^2(\mu)} \right).$$

(For further details see Appendix A.) Because $g^2$ is multiplied everywhere by $\beta_0$, Eq. (4) does not suffer from the above-mentioned problem; it defines an $N$-independent constant of dimension of mass. In what follows, it will be convenient to adopt a more general definition,

$$\Lambda_c = \mu (c \lambda(\mu))^{-\beta_1/\beta_0^2} \exp \left( -\frac{N}{\beta_0} \frac{1}{\lambda(\mu)} \right),$$

(3)
where the constant $c$ has a finite large-$N$ limit around which it can be expanded. In the standard definition $c = \beta_0/(2N)$, cf. Eq. (4). For the time being we will keep $c$ as a free parameter, and will discuss the sensitivity of our results to the choice of $c$ later. In a similar manner we introduce RGI bifermion operators as

$$\langle \bar{\Psi} \Psi \rangle_{\tilde{c}} \equiv N^{-2} \left( \tilde{c} \lambda(\mu) \right)^{\gamma/\beta_0} \langle \bar{\Psi} \Psi \rangle,$$

where $\tilde{c}$, like $c$, has a smooth large-$N$ limit. Its impact on $\langle \bar{\Psi} \Psi \rangle_{\tilde{c}}$ will be discussed later. Equation (6) will be applied both to the gluino condensate in SYM theory and to the quark condensate in the orientifold theory. With the above definitions the condensates and $\Lambda$’s approach finite limits as $N \to \infty$. Moreover, the ratio

$$R(N) = \frac{\langle \bar{\Psi} \Psi \rangle_{\tilde{c}}}{\Lambda_{\tilde{c}}^3}$$

also approaches a finite limit at large $N$ in both theories, and enjoys a smooth $1/N$ expansion.

### 3 The gluino condensate and the large-$N$ limit of the orientifold condensate

In SYM theory, where a number of exact results were obtained, the general considerations of Sect. 2 simplify considerably. First of all, the expression for $\Lambda$ in Eq. (5) becomes exact \[8\] rather than the two-loop approximation,

$$\Lambda_{\text{SYM}}^3 = \mu^3 \left( \frac{1}{c \lambda(\mu)} \right) \exp \left( - \frac{1}{\lambda(\mu)} \right).$$

(In this case the standard value of $c = 3/2$.) Furthermore, the definition (6) of the gluino condensate becomes

$$\langle \lambda \lambda \rangle_{\tilde{c}} \equiv \frac{1}{N^2} \left( \tilde{c} \lambda(\mu) \right) \langle \lambda^{a,\alpha} \lambda_{a}^{\alpha} \rangle.$$

Note that we deal here with the holomorphic part, there is no complex conjugate term in the right-hand side. The reason is that the exact results for

\[1\]The kinetic term of the fermion fields is canonically normalized, i.e. $\mathcal{L} = \bar{\Psi} \not\!{D} \Psi$. 

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the gluino condensate are routinely presented in terms of the holomorphic condensate, see below. Generally speaking, the VEV \( \langle \lambda^{a,\alpha}\lambda_{a}^{\alpha} \rangle \) is complex. We will assume the vacuum angle \( \theta \) to vanish. Then one can choose the vacuum state in such a way that \( \langle \lambda^{a,\alpha}\lambda_{a}^{\alpha} \rangle \) is real. We will discuss shortly how the gluino condensate defined in this particular way is mapped onto the orientifold theory.

The exact expression for the gluino condensate in SU(\( N \)) supersymmetric gluodynamics can be obtained from weak coupling considerations [9]. All numerical factors are carefully collected for SU(2) in the review paper [10]. A weak coupling calculation for SU(\( N \)) with arbitrary \( N \) was carried out in [11]. Note, however, that an unconventional definition of the scale parameter \( \Lambda \) is used in Ref. [11]. One can pass to the conventional definition of \( \Lambda \) either by normalizing the result to the SU(2) case [10] or by analyzing the context of Ref. [11]. Both methods give the same result, see Appendix A. When expressed in terms of our \( \Lambda_{\text{SYM}} \) it gives, for the ratio defined in Eq. (7)

\[
R_{\text{SYM}}(N) = -\frac{c\bar{c}}{2\pi^2}.
\]

This result is exact, there are no \( 1/N \) corrections. Note that all existing calculations of the gluino condensate were done in the Pauli-Villars (PV) regularization scheme. Equation (10), as it is, holds in that scheme. However, as we explain in Appendix B, for SUSY gluodynamics with dimensional reduction the PV scheme gives the same result as the more currently used MS scheme. Thus, all results following from (10) can be viewed as referring to the \( \overline{\text{MS}} \) scheme.

Now let us turn to the orientifold theory which is planar equivalent to supersymmetric gluodynamics. The first question to ask is the mapping of \( \langle \lambda^{a,\alpha}\lambda_{a}^{\alpha} \rangle \) onto \( \langle \bar{\Psi}_{[ij]}\Psi^{[ij]} \rangle \). There are many ways to establish a proper normalization. The simplest way is the comparison of the corresponding mass terms. The Dirac fermion \( \Psi \) of the orientifold theory can be replaced by two Weyl spinors, \( \xi_{[ij]} \) and \( \eta^{[ij]} \), so that the fermion mass term becomes:

\[
m\bar{\Psi}\Psi = m\xi\eta + \text{h.c.},
\]

while in softly broken SYM the mass term has the form

\[
\frac{m}{2}\lambda\lambda + \text{h.c.}
\]
Thus,
\[ \frac{1}{2} \langle \lambda \lambda \rangle \leftrightarrow \langle \xi \eta \rangle, \quad \text{or} \quad \langle \lambda \lambda \rangle \leftrightarrow \langle \bar{\Psi} \Psi \rangle \quad \text{at} \quad \theta = 0. \] (13)

The same identification is obtained from comparison of the two-point functions in the scalar and/or pseudoscalar channels in both theories.

In the orientifold theory
\[ \frac{\gamma}{\beta_0} = \frac{(1 - \frac{2}{N})(1 + \frac{1}{N})}{1 + \frac{4}{9N}} = 1 + O(1/N), \] (14)

and
\[ \frac{3\beta_1}{\beta_0^2} = 1 + \frac{49}{9N} - \frac{4}{3N^2} = 1 + O(1/N). \] (15)

The nonperturbative planar equivalence [2] implies
\[ R_{\text{OR}}(N) = R_{\text{SYM}} \tilde{K}(1/N), \quad \tilde{K}(1/N) = 1 + O(1/N), \] (16)

where the \( O(1/N) \) terms in \( \tilde{K} \) reflect deviations from the SYM/OR equivalence at non-planar level.

Expressing the result in terms of the conventional fermion bilinear through Eqs. (6) and (10) we arrive at
\[ \langle \bar{\Psi}_{[ij]} \Psi^{[ij]}(\mu) \rangle = -\frac{N^2}{2\pi^2} \mu^3 \left( \lambda(\mu) \right)^{-\frac{1}{2}(\gamma/\beta_0) - \frac{1}{2}(\beta_1/\beta_0^2)} \exp \left( -\frac{3N}{\beta_0} \lambda(\mu) \right) \]
\[ \times \tilde{K}(1/N) c^{1-(3\beta_1/\beta_0^2)} \bar{c}^{1-(\gamma/\beta_0)} \] (17)

where all quantities refer to those of QCD_{\text{OR}}, see Eqs (14) and (15).

This is our final general result. It shows a dependence on the choice of \( c \) and even more so on \( \bar{c} \) (since its exponent is of order \( 1/2 \)). Such dependence can be absorbed, however, in the definition of the factor \( \tilde{K}(1/N) \) modifying just the sub-leading terms. We, thus, rewrite (17) in a simpler form
\[ \langle \bar{\Psi}_{[ij]} \Psi^{[ij]}(\mu) \rangle = -\frac{N^2}{2\pi^2} \mu^3 \left( \lambda(\mu) \right)^{-\frac{1}{2}(\gamma/\beta_0) - \frac{1}{2}(\beta_1/\beta_0^2)} \exp \left( -\frac{3N}{\beta_0} \lambda(\mu) \right) K(1/N), \]
\[ K(1/N) = 1 + O(1/N). \] (18)
4 Finite-$N$ corrections and numerical results

The fact that QCD goes into YM theory at $N = 2$ implies the vanishing of the fermion condensate at $N = 2$. In other words we know for sure that the function $K(1/N)$ (as well as the previously introduced $\tilde{K}(1/N)$) must have a zero at $N = 2$. Moreover, arguments can be given that this zero is of the first order. Then we can write

$$K(1/N) = \left(1 - \frac{2}{N}\right) K^\ast(1/N),$$

where $K^\ast(1/N)$ is supposed to be free from “large” $1/N$ corrections. Assuming that $K^\ast(1/3)$ differs from 1 by $\pm 30\%$ at most, and setting $N = 3$, we arrive at the final formula for the quark condensate in one-flavor QCD

$$\frac{\langle \bar{\Psi}_{[ij]} \Psi^{[ij]}(\mu) \rangle}{\mu^3} = -\frac{3}{2\pi^2} K^\ast(1/3) \left(\frac{\lambda(\mu)}{\alpha_s(\mu)}\right)^{-\frac{1578}{961} \lambda(\mu)} \exp\left(-\frac{27}{31} \lambda(\mu)\right),$$

where $\lambda(\mu) = 3\alpha_s(\mu)/2\pi$, see Eq. (3).

As has been already mentioned, we expect non-planar corrections in $K^\ast$ to be in the ballpark $\pm 1/N$. If so, three values for $K^\ast(1/3)$,

$$K^\ast(1/3) = \{2/3, 1, 4/3\}$$

give a representative set. The only thing we need now is the value of $\lambda(\mu)$. Given $\mu$, $\lambda_{\overline{\text{MS}}}(\mu)$ and Eq. (21) one can get a numerical evaluation of the predicted quark condensate in one-flavor QCD.

The problem is that one-flavor QCD is different both from real QCD, with three massless quarks, and from quenched QCD in which lattice measurements have been recently carried out [12]. In quenched QCD there are no quark loops in the running of $\alpha_s$; thus, it runs steeper than in one-flavor QCD. On the other hand, in three-flavor QCD the running of $\alpha_s$ is milder than in one-flavor QCD.

To estimate the input value of $\lambda_{\overline{\text{MS}}}(\mu)$ we resort to the following procedure. First, starting from $\alpha_s(M_\tau) = 0.31$ (which is close to the world average) we determine $\Lambda_{\overline{\text{MS}}}^{(3)}$. Then, with this $\Lambda$ used as the input, we evolve the coupling constant back to 2 GeV according to the one-flavor formula. In this way we obtain

$$\lambda(2 \text{ GeV}) = 0.115.$$
Then
\[
\langle \bar{\Psi} \Psi \rangle = -\{0.014, 0.021, 0.028\} \text{ GeV}^3, \quad \mu = 2 \text{ GeV},
\]  

(23)
corresponding to three values of \( K_* \) in Eq. (21).

A check exhibiting the sensitivity of our prediction to the value of \( \lambda(2 \text{ GeV}) \) is provided by lattice measurements. Using the results of Ref. [13] referring to pure Yang-Mills theory one can extract \( \alpha_s(2 \text{ GeV}) = 0.189 \). (Here and below everything is in \( \overline{\text{MS}} \).) Then, as previously, we find \( \Lambda(0)_{\overline{\text{MS}}} \), and evolve back to 2 GeV according to the one-flavor formula. The result is
\[
\lambda(2 \text{ GeV}) = 0.097.
\]

(24)
The estimate (24) is smaller than (22) approximately by one \( \sigma \). This is natural since the lattice determinations of \( \alpha_s \) lie on the low side, within one \( \sigma \) of the world average. Using Eq. (24) we would get then
\[
\langle \bar{\Psi} \Psi \rangle = -\{0.05, 0.07, 0.09\} \text{ GeV}^3, \quad \mu = 2 \text{ GeV},
\]

(25)
Now we have to compare our prediction with an “empiric” value of the quark condensate in one-flavor QCD. Chiral perturbation theory allows one to determine the quark masses (see e.g. [14]). The Gell-Mann-Oakes-Renner (GMOR) relation [15] then implies
\[
\langle \bar{\Psi} \Psi \rangle = -0.015 \pm 0.005 \text{ GeV}^3, \quad \mu = 2 \text{ GeV}.
\]

(26)
One should remember that the very basis of this derivation, the GMOR relation, implies three light flavors. One can hope, though, that this particular quantity, \( \langle \bar{\Psi} \Psi \rangle \), is not very sensitive to the number of light flavors, although it is difficult to assign any uncertainty associated with the \( N_f \) dependence.

Lattice measurements of \( \langle \bar{\Psi} \Psi \rangle \) were performed [12] in quenched QCD. Two methods were used. The first determination was based on the measurement of the strange quark mass through a fit of the \( K \)-meson mass to its empiric value. Then \( \langle \bar{\Psi} \Psi \rangle \) was extracted from the GMOR relation. The second determination was a direct measurement of \( \langle \bar{\Psi} \Psi \rangle \). Both methods agree as far as the central value is concerned, while the uncertainties are much larger in the latter method. We quote here the result [12] obtained in the first method,
\[
\langle \bar{\Psi} \Psi \rangle = -0.019 \pm 0.004 \text{ GeV}^3, \quad \mu = 2 \text{ GeV}.
\]

(27)
If, instead, one uses Eq. (47) of Ref. [12] and substitutes there $a$ in physical units from Ref. [13], one gets

$$\langle \bar{\Psi} \Psi \rangle = -0.012 \pm 0.004 \text{ GeV}^3, \quad \mu = 2 \text{ GeV}. \quad (28)$$

In view of the above, it is not unreasonable to assume that the quark condensate in one-flavor QCD lies between these values. Our educated guess is

$$\langle \bar{\Psi} \Psi \rangle_{\text{one-flavor QCD}} = -0.016 \pm 0.005 \text{ GeV}^3, \quad \mu = 2 \text{ GeV}. \quad (29)$$

Comparison with Eq. (23) exhibits a significant overlap! Given all uncertainties involved in our numerical estimates, both from the side of supersymmetry/planar equivalence and the “empiric” side, we can state with satisfaction that the agreement is quite remarkable.

5 Conclusion

We started from supersymmetric gluodynamics where powerful methods, such as holomorphy, allow one to exactly calculate the gluino condensate. We then applied the non-perturbative planar equivalence obtained in [2] in conjunction with the orientifold large-$N$ expansion [3] to predict the value of the quark condensate in one-flavor massless QCD, up to subleading $1/N$ corrections. This seemingly first \textit{quantitative} application of a $1/N$ expansion in $D = 4$ produces a value for the quark condensate in remarkable agreement with the “empirical value.” Hopefully, this may pave the way to a whole new line of research based on translating a variety of exact results in supersymmetric theories to ordinary one-flavor QCD.

Two extensions of our method look worth being considered:

- Evaluation of subleading $1/N$ corrections, for which we were only able to give rough estimates here;

- Extension of our method in the direction of connecting non-supersymmetric or $\mathcal{N} = 1$ supersymmetric theories to $\mathcal{N} \geq 2$ theories for which even more is known.

Whether or not either one of these developments can be carried out remains to be seen.
ACKNOWLEDGMENTS

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Appendix A. Master formulae

In this Appendix we present basic formulae which are repeatedly used as an input in the bulk of the paper.

The master formula for the gluino condensate in SU($N$) supersymmetric gluodynamics is

$$\langle \lambda_\alpha^a \lambda^a \rangle = -32\pi^2 M_{PV}^3 \frac{1}{g^2} \exp\left(-\frac{8\pi^2}{Ng^2}\right)$$

$$= -4N M_{PV}^3 \frac{1}{\lambda} e^{-1/\lambda} ,$$

(A.1)

(A.2)

where $M_{PV}$ is the mass of the Pauli-Villars regulator, $g^2$ is the coupling constant at $M_{PV}$, the gluino field is normalized in such a way that the fermion part of the Lagrangian is (assuming that the vacuum angle $\theta = 0$)

$$L_{ferm} = \frac{i}{g^2} \bar{\lambda}^\alpha \not{D} \lambda^\alpha.$$

(A.3)

For numerical comparisons we need to know $\Lambda_{QCD}^{one-fl}$. This quantity is estimated in a number of ways in Sect. 4. We use the standard formula [6] for the running gauge coupling constant at two loops,

$$\alpha(\mu) = \frac{4\pi}{\beta_0 \ln \frac{\mu^2}{\Lambda^2}} \left(1 - \frac{2\beta_1}{\beta_0^2} \ln \frac{\mu^2}{\Lambda^2} + \ldots\right).$$

(A.4)

The corresponding expression for $\lambda$ is quoted in Eq. (4).
Appendix B. The Pauli-Villars vs. $\overline{\text{MS}}$ regularization schemes

All existing calculations of the gluino condensate are performed in the Pauli-Villars scheme, while all perturabative calculations and experimental determinations are routinely carried out in the $\overline{\text{MS}}$ scheme. Therefore, for our purposes it is necessary to know the relations between the corresponding $\alpha$’s or $\Lambda$’s.

The first derivation of this relation can be found in ’t Hooft’s pioneering paper [16], see Sect. 13. Unfortunately, the key expression (13.7) contained an error which, unfortunately, propagated in part in some reviews, e.g. Ref. [17]. It was corrected by Hasenfratz and Hasenfratz [18], see also Ref. [19], as well as in a later reprint of Ref. [16] (see [20]). In pure Yang-Mills theory

$$\Lambda_{\text{PV}} = \Lambda_{\overline{\text{MS}}} \exp \left( \frac{1}{22} \right).$$

The difference is entirely due to the fact that the vectorial index $\mu$ of the gauge connection $A^{a}_{\mu}$ takes $D$ rather than four values in $D$ dimensions. In QCD with $N_f$ flavors $1/22$ must be replaced by $\left( 22 - 4 \frac{N_f}{D} \right)^{-1}$.

In supersymmetric theories the situation is slightly more complicated on the one hand, and considerably simpler, on the other. Indeed, dimensional regularization per se cannot be used since it breaks the balance between the number of the fermionic and bosonic degrees of freedom. For instance, in SUSY gluodynamics the standard dimensional regularization would effectively imply $D - 2$ bosonic degrees of freedom and 2 fermionic.

The problem is fixed by using dimensional reduction with the subsequent application of the $\overline{\text{MS}}$ procedure. The supersymmetry is maintained because even in $D \neq 4$ dimensions the numbers of the fermionic and bosonic degrees of freedom match.

The most crucial point can be expressed as follows. In the ’t Hooft language the difference between the PV and $\overline{\text{MS}}$ schemes comes entirely from the non-zero mode parts of the determinants (quantum corrections in the instanton background). If one uses a more straightforward perturbative language, one can split the calculation of the gauge coupling renormalization (by virtue of the background field method, in a weak background) in two parts — the
one associated with the magnetic interaction with the background field, and
the one associated with the charge interaction. It is easy to see that the
magnetic part produces no difference between PV and MS. The difference is
entirely due to the charge part, which is in one-to-one correspondence with
the non-zero mode parts of the determinants in the instanton calculation.

For non-supersymmetric theories one must carry out a special dedicated
calculation to analyze the difference between PV and MS. In supersymmetric
theories the charge-interaction part in the gauge coupling renormalization
cancels (by the same token all non-zero mode determinants in the instanton
background cancel). This cancellation is due to the balance between the
number of fermionic and bosonic degrees of freedom. Thus, dimensional
reduction plus MS procedure give rise to the same $\Lambda$ as the Pauli-Villars
regularization,

$$\Lambda_{\text{PV}} = \Lambda_{\text{MS}}, \quad \text{SUSY dimensional reduction.} \quad (B.2)$$

References

[1] The gluino condensate in supersymmetric gluodynamics was first con-
jectured, on the basis of the value of his index, by E. Witten, Nucl. Phys.
B 202, 253 (1982). It was confirmed in an effective Lagrangian approach
and exactly calculated (by using holomorphy and analytic continuations
in mass parameters) in M. A. Shifman and A. I. Vainshtein, Nucl. Phys.

supersymmetric large N orientifold field theories*, hep-th/0302163 [Nucl.
Phys. B, in press].

QCD from a new 1/N expansion*, hep-th/0307097.

of the International Workshop on Supersymmetry and Unification of
Fundamental Interactions (SUSY 95), Eds. I. Antoniadis and H. Videau,


[20] G. ’t Hooft, in *Instantons in Gauge Theories*, Ed. M. Shifman (World Scientific, Singapore, 1994), p. 70. Note that Eq. (13.7) of the original paper becomes Eq. (13.6) in the reprint, while Eq. (13.7) of the reprint still contains a typo, -1 on the right-hand side should be replaced by -1/2. This misprint has no impact on Eq. (13.8) of the reprinted article.