Appendix

A. The Dirac Delta Function and the Normalisation of the Wavefunction of a Free Particle in Unbounded Space

The English physicist Dirac introduced a function which is extremely useful for many purposes of theoretical physics and mathematics. Precisely speaking, it is a generalised function which is only defined under an integral. We shall first give its definition, and then discuss its uses. The delta function (\( \delta \) function) is defined by the following properties (\( x \) is a real variable, \( -\infty \leq x \leq \infty \)):

1) \[ \delta(x) = 0 \quad \text{for} \quad x \neq 0 , \]  
   \hspace{1cm} (A.1)

2) \[ \int_a^b \delta(x) \, dx = 1 \quad \text{for} \quad a < 0 < b . \]  
   \hspace{1cm} (A.2)

The \( \delta \) function thus vanishes for all values of \( x \neq 0 \), and its integral over every interval which contains \( x = 0 \) has the value 1. The latter property means, speaking intuitively, that the \( \delta \) function must become infinitely large at \( x = 0 \). The unusual properties of the \( \delta \) function become more understandable when we consider it as the limiting case of functions which are more familiar. Such an example is given by the function

\[ \frac{1}{\sqrt{\pi u}} \exp\left(-\frac{x^2}{u^2}\right) , \]  
   \hspace{1cm} (A.3)

which is shown in Fig. A.1.

![Fig. A.1. The function \((1/\sqrt{\pi u}) \exp(-x^2/u^2)\) plotted against the variable \(x\). If we let the parameter \(u\) become smaller, the value of the function at \(x = 0\) gets larger and larger and the decrease to both sides gets steeper, until the function has finally pulled itself together into a \(\delta\) function.](image-url)
If we allow \( u \) to go to zero, the function becomes narrower and higher until it is finally just a "vertical line". We thus have

\[
\text{for } x \neq 0: \lim_{u \to 0} \frac{1}{\sqrt{\pi u}} e^{-x^2/u} = 0.
\]  

(A.4)

On the other hand, one can find in any integral table the following fact:

\[
\int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi u}} e^{-x^2/u} \, dx = 1,
\]

(A.5)

independently of the value of \( u \). If the limit \( u \to 0 \) is calculated it becomes clear that because of (A.4), we can write the integral (A.5) with any finite limits \( a \) and \( b \) with \( a < 0 < b \) without changing its value. This is just the relation (A.2).

In many practical applications in quantum mechanics, the \( \delta \) function occurs as the following limit:

\[
\delta(x) = \lim_{u \to \infty} \frac{1}{\pi} \frac{\sin(ux)}{x}.
\]

(A.6)

The property (A.2) is found to be fulfilled when we take into account that

\[
\int_{-\infty}^{+\infty} \frac{1}{\pi} \frac{\sin(ux)}{x} \, dx = 1.
\]

(A.7)

The property (A.1) is not so obvious. To demonstrate it, one has to consider that for \( u \to \infty, x \neq 0 \), \( \sin(ux)/x \) oscillates extremely rapidly back and forth, so that when we average the function over even a small region, the value of the function averages out to zero (Fig. A.2).

The \( \delta \) function has, in particular, the following properties:
A. The Dirac Delta Function and the Normalisation of the Wavefunction

for a continuous function \( f(x) \),

\[
\int_a^b f(x) \delta(x-x_0) \, dx = f(x_0), \quad a < x_0 < b
\]  
(A.8)

For a function \( f(x) \) which is \( n \) times continuously differentiable,

\[
\int_a^b f(x) \delta^{(n)}(x-x_0) \, dx = (-1)^n f^{(n)}(x_0), \quad a < x_0 < b
\]  
(A.9)

holds. Here \( f^{(n)} \) and \( \delta^{(n)} \) mean the \( n \)th derivatives w.r.t. \( x \). The proof of (A.8) follows immediately from (A.1, 2). The proof of (A.9) is obtained by \( n \)-fold partial integration. Furthermore,

\[
\delta(cx) = \frac{1}{|c|} \delta(x), \quad c \text{ real} \tag{A.10}
\]

is valid. The relation (A.1) is seen to be fulfilled on both sides. If we insert (A.10) in (A.2), we find

\[
\int_a^b \delta(cx) \, dx,
\]

which, after changing variables using \( cx = x' \), becomes

\[
\int_{a'}^{b'} \frac{1}{|c|} \delta(x') \, dx', \quad a' \equiv b' \quad \text{for} \quad c \equiv 0,
\]

which is thus, according to (A.2), equal to \( 1/|c| \).

We now turn to the question of the normalisation of wavefunctions in unbounded space, where we can limit ourselves to the one-dimensional case without missing any essentials. We start with wavefunctions which are normalised in the interval \( L \),

\[
\psi_k(x) = (1/\sqrt{L}) e^{ikx}, \tag{A.11}
\]

for which we have the normalisation integral

\[
\int_{-L/2}^{L/2} |\psi(x)|^2 \, dx = 1. \tag{A.12}
\]

If we furthermore assume that \( \psi(x) \) is periodic, \( \psi(x+L) = \psi(x) \), the \( k \)'s must have the form

\[
k = \frac{2\pi n}{L}, \quad n = 0, \pm 1, \pm 2, \ldots . \tag{A.13}
\]

It is easy to convince oneself that
\[
\int_{-L/2}^{L/2} \psi_k(x) \psi_{k'}(x) \, dx = \delta_{k,k'} \quad \text{(A.14)}
\]
\[
= \begin{cases} 
1 & \text{for } k = k' \\
0 & \text{for } k \neq k'. 
\end{cases} 
\quad \text{(A.15)}
\]

To prove this relation, the integral must be computed, taking account of (A.13). The integral yields
\[
\frac{1}{L} \int_{-L/2}^{L/2} e^{-ikx + ik'x} \, dx = \frac{1}{iL(k' - k)} \left( \exp[i(k' - k)L/2] - \exp[-i(k' - k)L/2] \right). 
\quad \text{(A.16)}
\]

If we now abbreviate \( k' - k \) with \( \xi \) and \( L/2 \) with \( u \), we may write (A.16) in the form
\[
\sin(\xi u)/\xi u. 
\quad \text{(A.17)}
\]

This is, however, apart from the factor \( 2\pi/L \), just the function which appears in (A.6) on the right under the limit, if we identify \( \xi \) with \( x \). If we thus divide (A.16) by \( 2\pi/L \) and form \( \lim \), we obtain on the left side of (A.16)
\[
\frac{1}{2\pi} \lim_{L \to \infty} \int_{-L/2}^{L/2} \exp(-ikx + ik'x) \, dx, 
\quad \text{(A.18)}
\]
which we may also write somewhat differently:
\[
\int_{-\infty}^{+\infty} \left[ \frac{1}{\sqrt{2\pi}} e^{ikx} \right]^* \left[ \frac{1}{\sqrt{2\pi}} e^{ik'x} \right] \, dx. 
\quad \text{(A.19)}
\]

The right-hand side of (A.16), using (A.17) and (A.6), goes to \( \delta(k' - k) \). We thus finally obtain
\[
\int_{-\infty}^{+\infty} \psi_k^*(x) \psi_{k'}(x) \, dx = \delta(k' - k), 
\quad \text{(A.20)}
\]
where
\[
\psi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}. 
\quad \text{(A.21)}
\]

Equation (A.20) with (A.21) generalises the relation (A.14) [with (A.11)] to the case of wavefunctions without finite boundary conditions and thus to the corresponding case of continuous \( k \) values. As may be seen in all practical applications, the \( \delta \) function in (A.20) always occurs under further integrals over \( k \) or \( k' \) (or both), so that we have found a self-consistent formalism.

Let the wavefunctions in (A.21) depend not upon \( k \), but upon \( p = \hbar k \); then we must observe (A.10). In order to normalise the new wavefunctions
\[
\psi_p(x) = N e^{ipx/\hbar}
\]
B. Some Properties of the Hamiltonian Operator, Its Eigenfunctions and Its Eigenvalues

We write the time-independent Schrödinger equation in the form

$$\mathcal{H} \psi_n = E_n \psi_n$$  \hspace{1cm} (B.1)

with the Hamiltonian

$$\mathcal{H} = -\frac{\hbar^2}{2m_0} \nabla^2 + V(r), \quad V(r) \text{ real}.$$  

The $\psi_n(r)$ are square-integrable eigenfunctions with the eigenvalues $E_n$. Here, $\psi_n = 0$ is excluded. The eigenvalues $E_n$ may be discrete or they may be continuous.

In the following, we denote by $\psi_\mu$ and $\psi_\nu$ the wavefunctions on which the operator $\mathcal{H}$ can act. We can now easily read off the following properties:

a) $\mathcal{H}$ is a linear operator, i.e. the relation

$$\mathcal{H} (c_\mu \psi_\mu + c_\nu \psi_\nu) = c_\mu \mathcal{H} \psi_\mu + c_\nu \mathcal{H} \psi_\nu$$

holds, where $c_\mu$ and $c_\nu$ are some complex numbers. In particular, it follows from this that every linear combination of eigenfunctions of $\mathcal{H}$ with the same eigenvalue $E$ is itself an eigenfunction of $\mathcal{H}$ with the eigenvalue $E$.

b) $\mathcal{H}$ is Hermitian, i.e. the equation

$$\int \psi_\mu^*(r) [\mathcal{H} \psi_\nu(r)] dV = \int [\mathcal{H} \psi_\mu(r)]^* \psi_\nu(r) dV$$  \hspace{1cm} (B.2)

is valid. It follows from (B.2) that for the operator of the potential energy, $V^*(r) = V(r)$. For the kinetic energy operator, (B.2) can be proved by double partial integration, taking into account the fact that the wavefunctions vanish at infinity.

c) The eigenvalues $E_n$ are real. This is a consequence of (B.2), if one inserts for $\psi_\mu$ and $\psi_\nu$ the same eigenfunction $\psi_n$ and utilises (B.1).

d) Eigenfunctions with different eigenvalues are orthogonal.

We take the following scalar products (different eigenvalues belong to the functions $\psi_m$ and $\psi_n$):
We subtract (B.4) from (B.3) and use (B.2) and the property that the eigenvalues are real:

\[ 0 = (E_n - E_m) \int \psi_n^*(r) \psi_n(r) dV. \]

The second bracket indicates the orthogonality of the wavefunctions.

Furthermore, it can be shown that eigenfunctions with the same eigenvalue may always be chosen to be orthogonal by using appropriate linear combinations.

## C. Derivation of Heisenberg’s Uncertainty Relation

For our derivation, it is important to define the uncertainties \( \Delta x \) and \( \Delta p \) which occur in (7.24) in a mathematically precise fashion. To this end, we introduce the variances

\[ (\Delta x)^2 = (x - \bar{x})^2 \]

and

\[ (\Delta p)^2 = (p - \bar{p})^2 \]

where the bar refers to the quantum-mechanical expectation value. We can always transform the coordinates so as to obtain

\[ \bar{x} = \bar{p} = 0. \]

We can then assume

\[ (\Delta x)^2 = x^2 \]

and

\[ (\Delta p)^2 = p^2 . \]

With the definitions

\[ (\Delta x)^2 = \int \psi^* x^2 \psi \, dx \]

and

\[ (\Delta p)^2 = -\hbar^2 \int \psi^* \frac{d^2 \psi}{dx^2} \, dx \]

and

\[ \int \psi_m^*(r) [\mathcal{H} \psi_n(r)] dV = E_n \int \psi_m^*(r) \psi_n(r) dV, \]  
\[ \int [\mathcal{H} \psi_m(r)]^* \psi_n(r) dV = E_m^* \int \psi_m^*(r) \psi_n(r) dV. \]
we can investigate the size of the expression

\[
\frac{(\Delta x)^2}{(\Delta p)^2} .
\]  

(C.8)

For this purpose, we introduce an auxiliary integral:

\[
J(\xi) = \int \left| \xi x \psi + \frac{d\psi}{dx} \right|^2 dx \geq 0 ,
\]  

(C.9)

where \( \xi \) is a still-to-be-determined real number. Since the absolute value is greater than 0, the overall integral as given in (C.9) is also greater than 0. We multiply out the individual terms in the integrand and thus obtain:

\[
J(\xi) = \xi^2 \int x^2 |\psi|^2 dx + \int \left| \frac{d\psi}{dx} \right|^2 dx + \xi \int x \left( \frac{\partial \psi^*}{\partial x} - \psi^* \frac{\partial \psi}{\partial x} \right) dx
\]  

\[
= \frac{(\Delta x)^2}{(\Delta p)^2} \frac{(\Delta p)^2}{h^2} ,
\]  

(C.10)

where the meaning of the first two integrals is indicated beneath them in each case. The integrand of the last integral in (C.10) can be written as a perfect differential, so that we obtain for this integral

\[
\int x \frac{d}{dx} (\psi^* \psi) dx .
\]  

(C.11)

Then we carry out an integration by parts, leading to:

\[
x x|\psi|^2 \bigg|_\infty^\infty - \int x^* \psi dx .
\]  

(C.12)

Since the wavefunction must vanish at infinity, the first summation term in (C.12) vanishes, while the second yields 1 because of the normalization condition. With these results, (C.10) can be simplified to yield:

\[
J(\xi) = \xi^2 (\Delta x)^2 - \xi + \frac{1}{h^2} (\Delta p)^2 ;
\]  

(C.13)

this expression is positive for all real values of \( \xi \). We now determine the value of \( \xi \) for which \( J \) is a minimum:

\[
J(\xi_{\text{min}}) \geq 0 .
\]  

(C.14)

If we differentiate (C.13) with respect to \( \xi \), we obtain the condition for a minimum:

\[
2 \xi_{\text{min}} (\Delta x)^2 - 1 = 0 ,
\]  

(C.15)
from which we find immediately

$$\xi_{\text{min}} = \frac{\hbar}{2(\Delta x)^2}. \quad (C.16)$$

As can be readily shown, (C.16) in fact gives the true minimum of (C.14). If we now insert (C.16) into (C.13) and take (C.14) into account, we obtain:

$$-\frac{1}{4} \frac{1}{(\Delta x)^2} + \frac{1}{h^2} (\Delta p)^2 \geq 0 \quad (C.17)$$

and from this, the relation

$$\frac{(\Delta x)^2}{(\Delta p)^2} \geq \frac{h^2}{4}. \quad (C.18)$$

The right-hand side gives the smallest value which the left-hand side may assume. In writing the Heisenberg uncertainty relation, one usually proceeds in a somewhat looser manner, taking the square root formally on both sides of (C.18) and setting

$$\sqrt{((\Delta x)^2)}^{1/2} = \Delta x \quad (C.19)$$

and

$$\sqrt{((\Delta p)^2)}^{1/2} = \Delta p, \quad (C.20)$$

thus obtaining

$$\Delta x \Delta p \geq \frac{\hbar}{2}. \quad$$
Solutions to the Problems

2.1 b) \( N_A = 6.25 \cdot 10^{23} \text{ mol}^{-1} \)

2.2 \( r_{\text{He}} = 2.08 \cdot 10^{-10} \text{ m} = 2.08 \text{ Å} \)

2.3

<table>
<thead>
<tr>
<th>Pressure ( p ) [mbar]</th>
<th>1</th>
<th>( 10^{-2} )</th>
<th>( 10^{-4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of collisions</td>
<td>1706</td>
<td>17</td>
<td>0.17</td>
</tr>
</tbody>
</table>

2.4 Helium: \( r_{\text{He}} = 1.33 \text{ Å} \); mercury: \( r_{\text{Hg}} = 1.19 \text{ Å} \)

2.5 The Bragg Law leads to a lattice-plane spacing of \( d = 2.78 \text{ Å} \)

2.6 Debye-Scherrer arrangement; Bragg’s Law gives \( r = 16.1 \text{ cm} \) in first order \((n = 1)\) and \( r = 33.1 \text{ cm} \) in second order \((n = 2)\).

2.7 Bragg reflection in first order: \( \theta = 3.63° \). The neutron wavelength is \( \lambda = 2.023 \cdot 10^{-11} \text{ m} \).

2.8 \( r_x = 0.35 \text{ nm} \)

3.1 The orbital radius of a charged particle:

\[
r = \frac{p}{qB} \rightarrow \text{momentum filter for } q, B \text{ constant.}
\]

With \( p = \sqrt{2mE} \) (classical energy-momentum relationship), we have

\[
r = \frac{\sqrt{2E}}{qB} \sqrt{m} \rightarrow \text{mass filter for } q, B \text{ and } E \text{ constant.}
\]

3.2

<table>
<thead>
<tr>
<th></th>
<th>(^1\text{H}^+)</th>
<th>(^2\text{H}^+)</th>
<th>(^3\text{H}^+)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass ( m ) [kg]</td>
<td>( 1.67 \cdot 10^{-27} )</td>
<td>( 3.35 \cdot 10^{-27} )</td>
<td>( 5.02 \cdot 10^{-27} )</td>
</tr>
<tr>
<td>Radius ( r ) of the circular orbit [cm]</td>
<td>9.14</td>
<td>12.9</td>
<td>15.8</td>
</tr>
<tr>
<td>Deflection on the screen [cm]</td>
<td>14.6</td>
<td>9.39</td>
<td>7.46</td>
</tr>
</tbody>
</table>

3.3 Parabola method:

\[
Y = AX^2
\]

(1)

\[
\begin{array}{ccc}
\text{H}^+ & \text{H}_2^+ \\
\text{H}^+ & \text{H}_2^+ \\
\text{Mass } m [\text{kg}] & 1.673 \cdot 10^{-27} & 3.347 \cdot 10^{-27} \\
A \text{ in Eq. (1) } [1/\text{m}] & 65.27 & 130.58 \\
v (1000 \text{ V}) [\text{m/s}] & 4.376 \cdot 10^5 & 3.094 \cdot 10^5 \\
X (1000 \text{ V}) [\text{m}] & 1.751 \cdot 10^{-2} & 1.238 \cdot 10^{-2} \\
Y (1000 \text{ V}) [\text{m}] & 2.000 \cdot 10^{-2} & 2.000 \cdot 10^{-2} \\
v (4000 \text{ V}) [\text{m/s}] & 8.752 \cdot 10^5 & 6.188 \cdot 10^5 \\
X (4000 \text{ V}) [\text{m}] & 8.753 \cdot 10^{-3} & 6.188 \cdot 10^{-3} \\
Y (4000 \text{ V}) [\text{m}] & 5.000 \cdot 10^{-3} & 5.000 \cdot 10^{-3} \\
\end{array}
\]

If both positive and negative particles are injected, one obtains parabolas in the \((+x, +y)\) and \((-x, -y)\) quadrants.

3.4 Abundance ratio \( P_n \) of \(^{235}\text{U}\) to \(^{238}\text{U}\) after the \( n \)th separation step:

\[
P_n = P_0 q^n \text{ where } q = \frac{P_1}{P_0} = \frac{0.754/99.246}{0.72/99.28} \quad (1)
\]

\[
\begin{array}{ccc}
\text{Amount of } ^{235}\text{U} & 50\% & 99\% \\
\text{P}_n & 1 & 99 \\
\text{n in Eq. (1)} & 106 & 205 \\
\end{array}
\]

4.1 When the aluminium foil is replaced by a gold foil of the same thickness, \( Z \) and \( N \) in the Rutherford scattering formula (4.20) are changed, i.e.

\[
\frac{\left( \frac{dn}{dt} \right)_{\text{Al}}}{Z_{\text{Al}}^2 N_{\text{Al}}} = \frac{\left( \frac{dn}{dt} \right)_{\text{Au}}}{Z_{\text{Au}}^2 N_{\text{Au}}},
\]
One finds \( \left( \frac{dn}{dt} \right)_{Au} = 36.11 \cdot \left( \frac{dn}{dt} \right)_{Al} \), meaning that 36110 particles/second are measured.

### 4.2

From the Rutherford scattering formula (4.20) it follows that

\[
\left( \frac{dn}{dt} \right) (\theta) = \frac{\sin^4 \left( \frac{10^\circ}{2} \right)}{\sin^4 \left( \frac{\theta}{2} \right)} \left( \frac{dn}{dt} \right) (10^\circ)
\]

\[
= \frac{57.7}{\sin^4 \left( \frac{\theta}{2} \right)}
\]

<table>
<thead>
<tr>
<th>( \theta [^\circ] )</th>
<th>10</th>
<th>45</th>
<th>90</th>
<th>135</th>
<th>180</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \left( \frac{dn}{dt} \right) ) [1/s]</td>
<td>( \times 10^6 )</td>
<td>2690.5</td>
<td>230.8</td>
<td>79.2</td>
<td>57.7</td>
</tr>
</tbody>
</table>

#### 4.3

The kinetic energy of the proton is converted completely to potential energy as a result of the Coulomb interaction. The distance of closest approach \( a \) is given by

\[
a = \frac{e^2 Z}{4 \pi \varepsilon_0 E_0} = \frac{1.138 \cdot 10^{-7} \text{eVm}}{E_0}.
\]

\( E_0 = 1 \text{ MeV} \): \( a = 1.14 \cdot 10^{-13} \text{m} \)
\( E_0 = 10 \text{ MeV} \): \( a = 1.14 \cdot 10^{-14} \text{m} \)

The radius of a gold nucleus (cf. Sect. 4.2.5):

\[
R = (1.3 \pm 0.1) A^{1/3} \cdot 10^{-15} \text{m} = 7.6 \cdot 10^{-15} \text{m}
\]

i.e. the proton “touches” the nucleus for \( E_0 = 15 \text{ MeV} \).

#### 4.4

It follows from (4.12) that \( \theta = 2 \arccot \frac{b m v_0^2}{k} \)

\( = 12.49^\circ \).

#### 4.5

It follows from (4.12) that \( b = 2.16 \cdot 10^{-13} \text{m} \).

#### 4.6

Sketch (see also Fig. 4.7):

- Integration of (4.8) from infinity (Point A) to Point B, the closest approach, with

\[
\phi_B = \frac{180^\circ - \theta}{2} \quad \text{and} \quad v_{\perp B} = \frac{v_0 b}{r_B} \cos \phi_B
\]

yields the distance

\[
r_B = \frac{m v_0 \cos \phi_B - b^2}{k (1 - \cos \phi_B)} \quad ; \quad k = \frac{2 Z e^2}{4 \pi \varepsilon_0}.
\]

Equation (4.12) leads to

\[
r_B = \frac{k}{2 E_0} \left( 1 + \frac{1}{\sin \theta/2} \right) = r_{Al} + r_a.
\]

The radius of the Al nucleus is thus \( r_{Al} = 2.7 \cdot 10^{-15} \text{m} \).

- From energy conservation,

\[
E_0 = \frac{1}{2} m v_B^2 + \frac{k}{r_B} \quad ; \quad (1)
\]

from momentum conservation, \( v_0 b = r_B v_B \) \( (2) \)

Equation (4.12) \( b = \frac{k}{2 E_0} \cot \theta/2 \). \( (3) \)

Elimination of \( v_B \) and \( b \) from \((1-3)\) yields
\[
    r_B = \frac{k}{2E_0} \left( 1 + \frac{1}{\sin \theta/2} \right)
\]

c) hyperbolic orbits \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \).

From the sketch, we see that \( r_B = a+c = a+\sqrt{a^2+b^2} \); with \( b = (2Ze^2/4\pi\varepsilon_0mv_0^2) \cot \theta/2 \) (Equation 4.12) and \( a = b \tan \theta/2 \) we find

\[
    r_B = \frac{k}{2E_0} \left( 1 + \frac{1}{\sin \theta/2} \right)
\]

4.7 a) The Rutherford scattering formula for protons (cf. 4.20):

\[
    \frac{dn}{nd\Omega} = \frac{\frac{Z^2e^4DN}{(16 \pi\varepsilon_0)^2E_0^2\sin^4(\theta/2)}}{(16 \pi\varepsilon_0)^2E_0^2\sin^4(\theta/2)} (1)
\]

Integrating over the angle \( \phi \) \( (d\Omega = \sin \theta d\theta d\phi) \) gives

\[
    \frac{dn}{nd\theta} = \frac{\frac{Z^2e^4DN}{64 \pi\varepsilon_0^2E_0^2\sin^3(\theta/2)}}{64 \pi\varepsilon_0^2E_0^2\sin^3(\theta/2)} (2)
\]

i.e. \( E_0 = 5.62 \cdot 10^{-13} \) J = 3.51 MeV.

b) Setting

\[
    \frac{dn}{n} = N\frac{d\alpha(\theta)}{d\Omega}
\]

in (1), we obtain the differential cross section

\[
    \frac{d\alpha}{d\Omega} = 1.052 \cdot 10^{-27} \text{ m}^2 = 10.52 \text{ barn}.
\]

c) From (4.12), with \( k = k_p = Ze^2/4\pi\varepsilon_0 \), we obtain

\[
    b = 2.81 \cdot 10^{-14} \text{ m}.
\]

4.8 Electrons are not subject to the strong interaction, and would therefore exhibit no anomalous Rutherford scattering. Neutrons have no electric charge.

5.1 \( E_{\text{photon}} = \frac{hc}{\lambda} = m_{\text{ph}}c^2 \Rightarrow m_{\text{ph}} = \frac{h}{\lambda c} \)

5.2 \( p_{\text{photon}} = mc = \frac{h}{\lambda} = \frac{E_{\text{photon}}}{c} \)

\( E_{\text{photon}} = 1 \text{ eV} \rightarrow p_{\text{photon}} = 5.34 \cdot 10^{-28} \) kg m s\(^{-1}\)

\( \lambda = 12400 \text{ Å} = 1.24 \cdot 10^{-6} \) m

5.3 Mass reduction: \( m = \frac{E}{c^2} = \frac{Pt}{c^2} \)

\( = 3.504 \cdot 10^{-8} \) kg

5.4 a) \( E_{\text{kin,e^{-}}} = E_{\text{kin,e^+}} = \frac{1}{2}E_y - m_e c^2 = 0.489 \text{ MeV} \)

\( = 7.83 \cdot 10^{-14} \) J

b) \( E = mc^2 = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} \)

\[ \rightarrow \frac{v}{c} = \sqrt{1 - \left( \frac{m_0c^2}{E} \right)^2} = 0.86 \]

5.5 \( E_y = \frac{hc}{\lambda} = 1.136 \cdot 10^{-18} \) J = 7.09 eV.

This corresponds to the binding energy of the \( \text{O}_2 \) molecule.

5.6 Photon energy:

\( E_y = \frac{hc}{\lambda} = 3.31 \cdot 10^{-19} \) J = 2.07 eV .

Photon flux for a radiant power of \( P = 1.8 \cdot 10^{-18} \) W:

\[ \frac{dN}{dt} = \frac{P}{E_y} = 5.4 \text{ s}^{-1} \]

5.7

<table>
<thead>
<tr>
<th>Frequency ( \nu ) [Hz]</th>
<th>10(^3)</th>
<th>2.42 \cdot 10^{21}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Photon energy ( h\nu ) [eV]</td>
<td>4.14 \cdot 10^{-12}</td>
<td>10(^7)</td>
</tr>
<tr>
<td>Number of photons per m(^2) and second</td>
<td>1.51 \cdot 10^{30}</td>
<td>6.2 \cdot 10^{11}</td>
</tr>
</tbody>
</table>

5.8 Photon momentum \( p_y = m_yc = E_y/c. \)

The radiation pressure for perpendicular incidence and complete absorption:

\[ P_{\text{rad}} = \frac{\Delta p_y}{\Delta t \Delta A} = \frac{\Delta E_y}{c \Delta t \Delta A} = 4.67 \cdot 10^{-6} \) kg/m\(^2\)

\[ \rightarrow \frac{P_{\text{rad}}}{P_{\text{atm}}} = 4.61 \cdot 10^{-11} . \]

The force on an area of 1 m\(^2\):

\( F = P_{\text{rad}}A = 4.67 \cdot 10^{-6} \) N .
Upon complete reflection, the momentum transfer is twice as large; \( P_{\text{rad}} \) and \( F \) are doubled.

5.9 a) Momentum conservation:

\[
E_y = \frac{h \nu}{c} = \sqrt{2M_{\text{atom}}E_{\text{atom}}} \rightarrow \frac{h^2}{2M\lambda^2}.
\]

b) Mercury:

\[
E_{\text{atom}} = 1.02 \cdot 10^{-29} \text{ J} = 6.4 \cdot 10^{-11} \text{ eV}.
\]

From (7.29) the energy-time uncertainty relation:

\[
\Delta E \cdot \Delta t \approx h.
\]

With \( \tau = \Delta t = 10^{-8} \text{ s} \), we have

\[
\Delta E \approx 6.6 \cdot 10^{-26} \text{ J} \gg E_{\text{atom}}.
\]

\( \rightarrow \) No effect on the position of the line, reabsorption of the photon by other mercury atoms is possible.

c) Nickel:

\[
E_{\text{atom}} = 2.59 \cdot 10^{-18} \text{ J} = 16.2 \text{ eV}.
\]

With \( \tau = \Delta t = 10^{-14} \text{ s} \), we have

\[
\Delta E \approx 6.6 \cdot 10^{-20} \text{ J} \ll E_{\text{atom}}.
\]

\( \rightarrow \) Noticeable line shift, reabsorption of the \( \gamma \) quantum by other (stationary) Ni atoms is not possible.

5.10 The Stefan-Boltzmann law gives the power radiated by the sphere:

\[ P = \pi d^2 \sigma T^4 , \quad T = 487 \text{ K} \quad \text{or} \quad 214^\circ \text{C}. \]

Annual loss of mass

\[ \Delta m = \frac{\Delta E}{c^2} = \frac{Pt}{c^2} = 3.5 \cdot 10^{-8} \text{ kg}. \]

5.11 The intensity of radiation at the sun’s surface \( I_S \) is

\[ P = I_S 4 \pi R^2 = I_S 4 \pi R^2 \quad \text{and therefore} \]

\[ I_S = 6.4 \cdot 10^7 \text{ W/m}^2. \]

From the Stefan-Boltzmann law (5.3), we have for the temperature

\[ T = \sqrt[4]{\frac{I_S}{\sigma}} = 5800 \text{ K}. \]

Integration of the spectral energy density \( u(\nu, T) \, d\nu \) from (5.6) gives the total energy density in the interior of a black body:

\[ U = \int_{0}^{\infty} u(\nu) \, d\nu = \frac{8 \pi^5 k^4}{15 c^3 h^3} T^4 = 0.854 \text{ J/m}^3. \]

The energy radiated from the sun is generated in the sun’s core and transported outwards. Therefore, \( I_S \) gets larger and larger towards the interior, and \( T \) and \( U \) increase strongly.

5.12 Using Wien’s displacement law (5.4)

\[ \lambda_{\text{max}} T = 0.29 \text{ cm K} , \quad \lambda_{\text{max}} = 9.67 \mu\text{m}. \]

Monochromatic energy density (5.6)

\[ u(\nu, T) = \frac{8 \pi h \nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1} \]

\[ = 1.3 \cdot 10^{-19} \text{ Js/m}^3 \approx 0.8 \text{ eV s/m}^3. \]

5.13 \( h \nu = m v^2 / 2 + eV_A = 4 \text{ eV} , \quad \lambda = c / v = 310 \text{ nm} \]

5.14 a) Braking voltage from (5.28):

\[ V_{\text{max}} = \frac{h \nu}{e} - V_A = \frac{hc}{\lambda e} - V_A = 1.2 \text{ V}. \]

b) \( E_{\text{kin}}^{\text{max}} = eV_{\text{max}} = 1.2 \text{ eV} = 1.92 \cdot 10^{-19} \text{ J} \),

\[ v_{\text{max}} = \sqrt{\frac{2E_{\text{kin}}}{m}} = 6.5 \cdot 10^5 \text{ m/s}. \]

5.15 a) Assuming that each photon releases an electron, the number of electrons per s and area \( j_s \) is

\[ j_s = \frac{\Delta N}{\Delta t \cdot A} = \frac{I_A}{hc} = 6.04 \cdot 10^9 \text{ (s m}^2)^{-1}. \]

b) The energy absorbed per unit of surface area and unit time is
\[ \frac{\Delta W}{\Delta t A} = \frac{\Delta N \cdot eV_A}{\Delta t \cdot A} = j_s e V_A \]
\[ = 1.208 \cdot 10^{10} \text{ eV/m}^2 \text{s} \]
\[ E_{\text{kin}} = h \nu - eV_A = 1.77 \cdot 10^{-19} \text{ J} = 1.1 \text{ eV} \]

5.16 Table of values:

<table>
<thead>
<tr>
<th>( \lambda ) [nm]</th>
<th>( v ) [10^{14} \text{ Hz}]</th>
<th>( V ) [V]</th>
</tr>
</thead>
<tbody>
<tr>
<td>366</td>
<td>8.20</td>
<td>1.48</td>
</tr>
<tr>
<td>405</td>
<td>7.41</td>
<td>1.15</td>
</tr>
<tr>
<td>436</td>
<td>6.88</td>
<td>0.93</td>
</tr>
<tr>
<td>492</td>
<td>6.10</td>
<td>0.62</td>
</tr>
<tr>
<td>546</td>
<td>5.50</td>
<td>0.36</td>
</tr>
<tr>
<td>579</td>
<td>5.18</td>
<td>0.24</td>
</tr>
</tbody>
</table>

Equation (5.27): \( V = \frac{h \nu}{e} - V_A \).

Fitting using the least-squares method gives
\[ \frac{h}{e} = (4.111 \pm 0.012) \cdot 10^{-15} \text{ Js/C} \]
\[ V_A = (1.8939 \pm 0.0076) \text{ V} \]

a) \( V = 0 \): \( v_{\text{lim}} = \frac{V_A}{h/e} = (4.607 \pm 0.032) \cdot 10^{14} \text{ Hz} \)
b) \( W_A = eV_A = (3.034 \pm 0.012) \cdot 10^{-19} \text{ J} \)
c) \( h/e = (4.111 \pm 0.012) \cdot 10^{-15} \text{ Js/C} \)

5.17 Na D lines: \( \lambda_D \approx 589 \text{ nm} \)

\[ h \nu = \frac{hc}{\lambda} = eV \] with \( V = 2.11 \text{ V} \)
\[ h/e = \frac{V\lambda}{c} = 4.14 \cdot 10^{-15} \text{ Js/C} \]

5.18 Calculation of the kinetic energy of an electron from the orbital data:
\[ e v B = m_0 \frac{v^2}{R} \rightarrow E_{e-} = \frac{1}{2} m_0 v^2 = \frac{e^2 B^2 R^2}{2 m_0} \]
\[ \approx 7.9 \text{ keV} \]

From Problem 5.19 we have
\[ E_e = \Delta E = \frac{E(1 - \cos \theta)}{1 + \xi(1 - \cos \theta)} \]
\[ \xi = \frac{E}{m_0 c^2} \]

From (1) and (2) it follows that
\[ \xi = \frac{A + \sqrt{A^2 + 8A}}{2} \approx 0.184 \]
\[ A = \frac{e^2 B^2 R^2}{2 m_0 c^2} \]

→ Energy and wavelength of the incident photon are
\[ E = m_0 c^2 \xi = 94 \text{ keV} \]
\[ \lambda = h/c/E = 0.132 \text{ Å} \]

5.19 Energy and momentum conservation:
\[ E - E' = E_e \]
\[ \frac{E}{c} - \frac{E'}{c} \cos \theta = p_e \cos \theta \]
\[ \frac{E'}{c} \sin \theta = p_e \sin \theta \]

Relativistic energy-momentum relation:
\[ p_e^2 = \frac{1}{c^2} \left( E_e^2 + 2 m_0 c^2 E_e \right) \]

a) \((1)^2 + (2)^2 - (1)^2/c^2 \) and (4) yield
\[ E' = \frac{E}{1 + \xi(1 - \cos \theta)} \]
\[ \xi = \frac{E}{m_0 c^2} \]

thus, the energy loss of the photon is
\[ \Delta E = E - E' = E \frac{\xi(1 - \cos \theta)}{1 + \xi(1 - \cos \theta)} = 96.8 \text{ eV} \]

and the change in its frequency and wavelength is
\( \Delta v = \frac{\Delta E}{h} = 2.34 \cdot 10^{16} \text{ Hz} , \)
\( \Delta \lambda = \frac{h}{m_0 c} (1 - \cos \theta) = 1.21 \cdot 10^{-12} \text{ m} . \)

b) (2)\(^2\) + (3)\(^2\) and (5) yield
\[
p_e^2 = m_0^2 c^2 \frac{E^2 (1 - \cos \theta) [2 + \xi (2 + \xi) (1 - \cos \theta)]}{(1 + \xi (1 - \cos \theta))^2} .
\]
This gives for the electron’s momentum
\( p_e = 5.31 \cdot 10^{-24} \text{ kg m/s} , \)
and for its kinetic energy,
\( E_e = \Delta E_\gamma = 96.8 \text{ eV} . \)

Equations (2), (3) and (5) give
\[
cot \theta = (\xi + 1) \sqrt{\frac{1 - \cos \Theta}{1 + \cos \Theta}} ,
\]
leading to \( \theta = 59.5^\circ . \)

5.20 (Compare Problem 5.19)

a) \( \Delta \lambda = \frac{h}{m_0 c} (1 - \cos \theta) = 0.0243 \text{ Å} . \)

b) \( E_{\text{kin}} = \Delta E_\gamma = E \frac{(E/m_0 c^2)(1 - \cos \theta)}{1 + (E/m_0 c^2)(1 - \cos \theta)} ; \)
\( E = \frac{hc}{\lambda} ; \rightarrow E_{\text{kin}} = 293.7 \text{ eV} . \)

c) \( \frac{\Delta E_\gamma}{E} (1 \text{ Å, } 90^\circ) = \frac{293.7 \text{ eV}}{12409 \text{ eV}} = 2.37\% \)

d) \( \frac{\Delta E_\gamma}{E} (0.1 \text{ Å, } 90^\circ) = 19.5\% \)

5.21 \( \lambda = 355 \text{ nm}, n = 1.8 \cdot 10^{15} \)
\( P = 3.3 \cdot 10^7 \text{ Watt} \)
\( I = 200 \text{ Watt} \)

6.1 a) For levitation equilibrium
\( F^\uparrow = F_{\text{electr.}} = e \frac{V}{d} = \frac{4}{3} \pi r^3 q g = F_{\text{grav.}} = F^- , \)
\( (1) \)
\[
r = \sqrt[3]{\frac{3 \text{ eV}}{4 \pi d q g}} = 2.79 \cdot 10^{-7} \text{ m} .
\]

Motion after field reversal
\( F^\uparrow = F_{\text{visc.}} = 6 \pi \eta rv = e \frac{V}{d} + \frac{4}{3} \pi r^3 q g \)
\( = F_{\text{electr.}} + F_{\text{grav.}} = F^- , \)
\( (2) \)
\[
v = \frac{1}{6 \pi \eta r} \left( e \frac{V}{d} + \frac{4}{3} \pi r^3 q g \right)
= 1.67 \cdot 10^{-5} \text{ m/s} .
\]

b) The correction introduces an additional term
\( 1/(1 + \lambda A r) \) on the left side of (2). Eliminating \( r \)
from (1) and (2) yields
\[
e = \frac{\pi d q g}{6 V} \left( -A \lambda + \sqrt{A^2 \lambda^2 + \frac{9 \eta v}{q g}} \right)^{1/3} .
\]

6.2 Radius of the flight path \( r = 1.668 \text{ m} \)

Mass of the particle \( m = \frac{q^2 B^2 r^2}{2 E} \)
\( = 1.86 \cdot 10^{-28} \text{ kg} \)

Rest energy \( E = mc^2 = 104.5 \text{ MeV} . \)
The particle is most probably a muon.

6.3 b) The electrons leave the apparatus without deflection when the midpoints of both condensers are reached just at the moment of opposite zero-crossings of the high frequency field.

Possible frequencies:
\[
n = 0, 1, 2 \ldots .
\]
\[
\nu_n = \sqrt{\frac{e}{m} \left( n + \frac{1}{2} \right) \sqrt{\frac{2 V}{l}}} \]
\( n = 0, 1, 2 \ldots . \)

c) \( \nu_{\text{min}} = \nu_0 = \sqrt{2 V \left( \frac{e}{m} \right) \frac{1}{2l}} = 66.31 \text{ MHz} . \)
6.4 \( E_{\text{ges}} = E_0 + E_{\text{kin}} - \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} \); 
\( E_{\text{kin}} = E_0 = m_0 c^2 \) 
\( \rightarrow v = \frac{\sqrt{3}}{2} c \approx 0.866 c \)

6.5 \( \frac{m}{m_0} = \frac{E_0 + E_{\text{kin}}}{E_0} = 1 + \frac{E_{\text{kin}}}{m_0 c^2} = 2.96 \)

6.6 \( \lambda = \frac{h}{p} = \frac{h}{m_0 c} \frac{\sqrt{1 - v^2/c^2}}{v/c} = 1.818 \cdot 10^{-12} \text{ m} \)

6.7 Relativistic relationship \( \lambda(E_{\text{kin}}) \):
\[ \lambda = \frac{hc}{E_0} \frac{1}{\sqrt{K^2 + 2 K}} \quad ; \quad K = \frac{E_{\text{kin}}}{E_0} \]

Scattering angle \( \alpha \) in the 1st order from Bragg's Law:
\[ \lambda = 2 d \sin \frac{\alpha}{2} \]

<table>
<thead>
<tr>
<th>( E_{\text{kin}} ) [eV]</th>
<th>( 10^2 )</th>
<th>( 10^3 )</th>
<th>( 10^5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda ) [m]</td>
<td>( 1.227 \cdot 10^{-9} )</td>
<td>( 1.227 \cdot 10^{-10} )</td>
<td>( 3.877 \cdot 10^{-11} )</td>
</tr>
<tr>
<td>( \alpha ) [°]</td>
<td>33.15</td>
<td>10.35</td>
<td>0.99</td>
</tr>
</tbody>
</table>

For scattering angle \( \alpha = 30° \): \( E_{\text{kin}} = 121.4 \text{ eV} \).

6.8 \( \tilde{E}_{\text{kin}} \) (298 K) = \( \frac{3}{2} kT = 6.17 \cdot 10^{-21} \text{ J} = 38.5 \text{ meV} \)

\[ \lambda = \frac{h}{\sqrt{2 m \tilde{E}_{\text{kin}}}} = 1.458 \cdot 10^{-10} \text{ m} \]

Bragg reflection in 1st order at \( \theta = 15.0° \).

6.9 \( \lambda = \frac{h}{p} = \frac{h}{\sqrt{2 m E_{\text{kin}}}} \) (nonrelativistic)
\[ \lambda(a = 1 \text{ m}) = 3.232 \cdot 10^{-5} \text{ m} \]
\[ \lambda(a = 0.5 \cdot 10^{-10} \text{ m}) = 2.287 \cdot 10^{-10} \text{ m} \]

7.1 The normalisation constant \( N \) is
\[ N = (2 \Delta k)^{-1/2} \pi^{-3/4} \]

After Fourier transformation, one obtains the spatial wavefunction of the free particle for \( t > 0 \):
\[ \psi(x, t) = \frac{m_0 \Delta k}{\sqrt{\pi (m_0 + i h t \Delta k^2)}} \cdot \exp \left\{ -\frac{x^2 m_0 \Delta k^2}{2 (m_0 + i h t \Delta k^2)} \right\} \]

From it, the probability density for the particle can be calculated to be:
\[ |\psi(x, t)|^2 = \frac{m_0 \Delta k}{\sqrt{\pi (m_0^2 + h^2 t^2 \Delta k^4)}} \cdot \exp \left\{ -\frac{x^2 m_0^2 \Delta k^2}{(m_0^2 + h^2 t^2 \Delta k^4)} \right\} \]

The normalisation remains valid for all times, since every solution of a time-dependent Schrödinger equation remains normalised. This can be demonstrated in general by calculating the time dependence of the normalisation integral and using the fact that the wavefunctions obey the Schrödinger equation. If the wavefunction is known as an explicit function of \( x \) and \( t \), one could naturally calculate the normalisation integral for any arbitrary time; this method, however, demands considerably more calculational effort. The probability density is a Gaussian function having a width at arbitrary time \( t \) of:
\[ \Delta x(t) = \frac{1}{m_0 \Delta k} \sqrt{m_0^2 + h^2 t^2 \Delta k^4} ; \]

thus, the wave packet spreads out with time.

The factor \( N^2 = \exp(- (k/\Delta k)^2) \) is the probability density in \( k \)-space, i.e. it denotes the probability of finding the particle with a momentum = \( h k \) as a result of a momentum measurement.

7.2 \[ \Delta x = \frac{1}{(\Delta k)^2} + \frac{h^2 t^2 (\Delta k)^2}{m_0^2} \]

For \( t = 0 \) we have: \( \Delta k = 10^8 \text{ cm}^{-1} \)
\[ t = \frac{m_0}{\hbar \Delta k} \sqrt{(\Delta x)^2 - \frac{1}{(\Delta k)^2}} \]

\[ t \approx m_0 \cdot 1.5 \cdot 10^{32} \text{s} \]

\[ \rightarrow t \approx 1.3 \cdot 10^4 \text{s for an electron.} \]

7.3 In three dimensions, we use a trial wavefunction for the particle consisting of the product of the three one-dimensional wavefunctions from Problem 7.1:

\[ \psi(r, t) = \psi(x, t) \psi(y, t) \psi(z, t) \]

This is correct for arbitrary times, since the Hamiltonian is the sum of three one-dimensional operators, each of which commutes with the others:

\[ \mathcal{H} = -\frac{\hbar^2}{2m_0} \left( \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2} \right) \]

The three-dimensional Schrödinger equation thus is decomposed into three mutually independent, one-dimensional Schrödinger equations, whose solutions were already found in Problem 7.1.

Thus, the wave packet spreads out in each of the three spatial directions:

\[ \Delta x(t) = \frac{1}{m_0 \Delta k_x} \sqrt{m_0^2 + \hbar^2 t^2 \Delta k_x^2} \]

\[ \Delta y(t) = \frac{1}{m_0 \Delta k_y} \sqrt{m_0^2 + \hbar^2 t^2 \Delta k_y^2} \]

\[ \Delta z(t) = \frac{1}{m_0 \Delta k_z} \sqrt{m_0^2 + \hbar^2 t^2 \Delta k_z^2} \]

8.1 \[ E_{\text{kin}} = 1.387 \cdot 10^{-26} \text{J} = 8.657 \cdot 10^{-8} \text{eV} \]

\[ v = 4.07 \text{ms}^{-1} \]

8.2

<table>
<thead>
<tr>
<th>\lambda [\text{Å}]</th>
<th>\tilde{\nu} [\text{cm}^{-1}]</th>
<th>\tilde{\nu}/R_H</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>3669.42</td>
<td>27252.3</td>
<td>0.24848</td>
<td>&gt;20</td>
</tr>
<tr>
<td>3770.06</td>
<td>26524.8</td>
<td>0.24184</td>
<td>&gt;10</td>
</tr>
<tr>
<td>3835.40</td>
<td>26072.9</td>
<td>0.23772</td>
<td>9</td>
</tr>
<tr>
<td>3970.07</td>
<td>25188.5</td>
<td>0.22966</td>
<td>7</td>
</tr>
<tr>
<td>4340.47</td>
<td>23039.0</td>
<td>0.21006</td>
<td>5</td>
</tr>
</tbody>
</table>

8.3 Practically all the atoms are in their ground states before the absorption; one obtains the lines of the Lyman series

\[ E_{\gamma} = \hbar c \tilde{\nu} = \hbar c R_H \left( 1 - \frac{1}{n^2} \right) \]

\[ \lambda = \frac{1}{\tilde{\nu}} = \frac{1}{R_H \left( 1 - \frac{1}{n^2} \right)} ; \quad n = 2, 3, \ldots \]

8.4 a) Diffraction of mth order by a grating: \( \lambda = (d/m) \sin \theta \) gives the wavelength of the Balmer line, \( \lambda = 486.08 \text{nm} \) and the initial state level \( n = 4 \).

b) The Balmer lines from the \( n = 32 \) and \( n = 31 \) states must still be able to be separated, giving the necessary resolution \( A \):

\[ A = \frac{\tilde{\nu}(n = 32)}{\tilde{\nu}(n = 32) - \tilde{\nu}(n = 31)} = 3890 = m \cdot p \]

where \( m \) is the diffraction order and \( p \) the number of rulings of the grating; the latter must thus be at least 3890.

8.5 From Sect. 8.4 we have

\[ E_{\text{kin}} = \frac{a}{2r_n} ; \quad E_{\text{pot}} = -\frac{a}{r_n} ; \quad a = \frac{Ze^2}{4\pi\varepsilon_0} ; \]

therefore \( E_{\text{kin}} \neq |E_{\text{pot}}| \).

Total energy \( E_n = E_{\text{pot}} + E_{\text{kin}} = -(a/2r_n) \).

This amount of energy is released, e.g. by the emission of photons, when a Bohr atom is formed.

\[ \frac{E_{\text{pot}}}{E_{\text{kin}}} = -\frac{a/r_n}{a/2r_n} = -2 \quad \text{independently of } n. \]

8.6 a) \[ F_z = m \frac{v^2}{r} = G \frac{Mm}{r^2} \quad (1) \]

b) \[ E_{\text{kin}} = \frac{1}{2} mv^2 = \frac{1}{2} G \frac{Mm}{r} \]

c) \[ E_{\text{pot}} = -\frac{GMm}{r} \]
8.7 For a ring current:

\[ I = -\frac{e\omega}{2\pi} = -1.055 \times 10^{-3} \text{A} \cdot \frac{1}{n^3} \]

and the magnetic moment

\[ |\mu| = \frac{1}{2} e\omega r^2 = \frac{e}{2m_0} |I| = \frac{eh}{2m_0} n = \mu_B n \]

(Compare Sect. 12.2).

| \( n \) | \( I \text{ [A]} \) | \( |\mu| \text{ [Am}^2\text{]} \) |
|-------|---------|----------------|
| 1     | 1.055 \times 10^{-3} | 9.274 \times 10^{-24} |
| 2     | 1.318 \times 10^{-4} | 1.855 \times 10^{-23} |
| 3     | 3.905 \times 10^{-5} | 2.782 \times 10^{-23} |

8.8

\[ \omega = \frac{me^4}{64\pi^3 \varepsilon_0^2 \hbar^3} \cdot \frac{1}{n^3} = 3.288 \times 10^{15} \text{s}^{-1} \cdot \frac{1}{n^3} \]

\[ r = \frac{8\pi \varepsilon_0 \hbar^2}{me^2} \cdot n^2 = 1.059 \times 10^{-10} \text{m} \cdot n^2 \]

\[ E_n = -\frac{me^4}{64\pi^2 \varepsilon_0^2 \hbar^2} \cdot \frac{1}{n^2} = -6.80 \text{eV} \cdot \frac{1}{n^2} \]

where \( r \) is the distance between the \( e^- \) and \( e^+ \).
(For the calculation, see Sect. 8.4.)

8.9

a) \( E_n(\mu) = -\frac{Z^2 e^4 m_\mu}{32\pi^2 \varepsilon_0^2 \hbar^2} \cdot \frac{1}{n^2} = -2813 \text{eV} \cdot \frac{Z^2}{n^2} \)

b) \( r_n = \frac{4\pi \varepsilon_0 \hbar^2}{Ze^2 m_\mu} \cdot n^2 = 2.56 \times 10^{-3} \text{Å} \cdot \frac{n^2}{Z} \)

c) \( h \nu = E_2(\mu) - E_1(\mu) = 2110 \text{eV} \cdot Z^2 \)

8.10

With \( \omega_n \) from (8.9) and \( \Delta t = 10^{-8} \text{s} \), we have

\[ N = \frac{\Delta t}{2\pi} \cdot \omega = 6.583 \times 10^7 \cdot \frac{1}{n^3} . \]

a) \( n = 2 \) gives \( N = 8.228 \times 10^6 \);

\[ K = \frac{4.5 \times 10^9}{N} = 547 \]

b) \( n = 15 \) gives \( N = 1.950 \times 10^4 \);

\[ K = 230800 . \]

8.11 Pickering series: \( \tilde{\nu}_p = 4R_{\text{He}} \left( \frac{1}{4^2} - \frac{1}{n^2} \right) \)

Rydberg constant: \( R_{\text{He}} = \frac{R_{\infty}}{1 + \frac{m_0}{M_{\text{He}}} \cdot \frac{1}{n^2}} \)

1st line: \( n = 5 \); \( \Delta \tilde{\nu} = 0.443 \text{ cm}^{-1} \);

\[ \Delta E = 5.5 \times 10^{-5} \text{eV} \]

3rd line: \( n = 7 \); \( \Delta \tilde{\nu} = 0.828 \text{ cm}^{-1} \);

\[ \Delta E = 1.03 \times 10^{-4} \text{eV} . \]

8.12 Hydrogen-like atoms:

\[ \tilde{\nu} = R \left( \frac{1}{n^2} - \frac{1}{n^{'2}} \right) ; \quad n' < n . \]

R_H = 109677.581 cm\(^{-1}\) ;

R_{He} = 109722.398 cm\(^{-1}\) .

We find 400 nm < \( \lambda < 700 \text{ nm} \) for

Hydrogen with \( n = 3, 4, 5, 6 \) i.e. \( n' = 2 \)

Helium\(^+\) with \( n = 4 \)

\[ n = 6 \ldots 13 \quad n' = 4 \]

\[ n = 12 \ldots \infty \quad n' = 5 . \]
8.13 The relative deviation $\gamma$ between the Sommerfeld energy formula $E_{n,k}$ (8.29) and the Bohr energy formula $E_n$ in the hydrogen atom for $n = 2$:

$$\gamma = \frac{E_{2,k} - E_2}{E_2} = \frac{\alpha^2}{4} \left( \frac{2}{k} - \frac{3}{4} \right)$$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$1$</th>
<th>$2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>$1.67 \cdot 10^{-5}$</td>
<td>$3.33 \cdot 10^{-6}$</td>
</tr>
</tbody>
</table>

8.14 Wavelength $\lambda_2$ of the second laser

$$\lambda_2 = \frac{hc}{E_1(H) \left( \frac{1}{n^2} - 1 \right) - 11.5 \text{ eV}}$$

<table>
<thead>
<tr>
<th>$n_2$</th>
<th>$E_{n_2} [\text{eV}]$</th>
<th>$r_{n_2} [\text{Å}]$</th>
<th>$\lambda_2 [\text{nm}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>$-3.40 \cdot 10^{-2}$</td>
<td>212</td>
<td>603.5</td>
</tr>
<tr>
<td>30</td>
<td>$-1.51 \cdot 10^{-2}$</td>
<td>477</td>
<td>598.0</td>
</tr>
<tr>
<td>40</td>
<td>$-0.85 \cdot 10^{-2}$</td>
<td>848</td>
<td>596.1</td>
</tr>
<tr>
<td>50</td>
<td>$-0.54 \cdot 10^{-2}$</td>
<td>1325</td>
<td>595.3</td>
</tr>
</tbody>
</table>

Required linewidth $\Delta E \approx E_{51} - E_{50} \approx 2 \cdot 10^{-4} \text{ eV}$.

8.15 a) $v_n = \frac{\omega_n}{2 \pi} = \frac{e^4 m_0}{32 \pi^3 \epsilon_0^2 h^3 n^3} = \frac{6.58 \cdot 10^{15} \text{ s}^{-1}}{n^3}$

b) $v_{n \rightarrow n-1} = \frac{e^4 m_0}{32 \pi^3 \epsilon_0^2 h^3 n^3} \left[ \frac{1 - \frac{2}{n}}{1 - \frac{2}{n} + \frac{1}{n^2}} \right]$

c) $\lim_{n \rightarrow \infty} v_{n \rightarrow n-1} = v_n$

(compare Sect. 8.11: correspondence principle)

8.16 The motion of the nucleus can be taken into account by replacing the electron's mass $m_0$ with the reduced mass

$$\mu = m_0 M/(m_0 + M) \quad (M = \text{nuclear mass})$$

$$E_n(A, Z) = \frac{E_1(H)}{n^2} \frac{1}{1 + \frac{1}{\mu} \cdot \frac{m_0}{m_p}} Z^2 ;$$

($A = \text{atomic mass number}$).

$$m_0 = \frac{1}{1836.15}$$

<table>
<thead>
<tr>
<th>Atom</th>
<th>$\Delta E(Z, n)$</th>
<th>$E(Z, n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^1\text{H}$</td>
<td>$5.45 \cdot 10^{-4}$</td>
<td></td>
</tr>
<tr>
<td>$^2\text{H}$</td>
<td>$2.75 \cdot 10^{-4}$</td>
<td></td>
</tr>
<tr>
<td>$^3\text{H}$</td>
<td>$1.82 \cdot 10^{-4}$</td>
<td></td>
</tr>
<tr>
<td>$^4\text{He}^+$</td>
<td>$1.36 \cdot 10^{-4}$</td>
<td></td>
</tr>
<tr>
<td>$^7\text{Li}^{2+}$</td>
<td>$0.78 \cdot 10^{-4}$</td>
<td></td>
</tr>
</tbody>
</table>

8.17 $n = 1 \rightarrow n = 2$ ; $n = 1 \rightarrow n = 3$

8.18 $n = 3, 4, 5, 6$; plot as a function of $n^{-2}$.

8.19 $E_n = E_n(\text{Bohr}) \cdot \mu/m_0 e^2$. Spectrum: similar to the Balmer series.

9.1 The Schrödinger equation of the force-free particle is:

$$i \hbar \frac{\partial}{\partial t} \psi(x, t) = -\frac{\hbar^2}{2 m_0} \frac{\partial^2}{\partial x^2} \psi(x, t) .$$

Both sides of the equation can be calculated by exchanging the differentiations with the integrations in $k$-space. Substituting the given dispersion relation for the free particle then demonstrates the equality of the left and the right sides.

9.2 a) By substituting $\psi(r, t)$ into (9.32) .

b) We require $\int \psi^\ast \psi dV = 1$.

Since $\int \phi^\ast \phi dV = \delta_{jk} , \quad |C_1|^2 + |C_2|^2 = 1$ must hold ,
or, in general, $\sum_j |C_j|^2 = 1$.

9.3 From the requirement that the wavefunction be normalisable, we find from the time-independent Schrödinger equation:

$$\psi(x) = \begin{cases} \frac{N}{\sqrt{A^3}} \exp(kx) , & x \leq 0 \\ \frac{N}{\sqrt{A^3}} \exp(-kx) , & x \geq 0 \end{cases}$$

with $k = \sqrt{-\frac{2 m_0 E}{\hbar^2}}$.
The normalisation factor of the exponential function must be the same on both sides, in order for the wavefunction to remain continuous at \( x = 0 \). The jump condition is

\[
\psi' (\varepsilon) - \psi' (-\varepsilon) = - \frac{2m_0\beta}{\hbar^2} \psi(0) .
\]

From this, one finally obtains \( k = \frac{m_0\beta}{\hbar^2} \), and the energy eigenvalue \( E = -(\frac{m_0\beta^2}{2\hbar^2}) \). The normalisation factor is \( N = \sqrt{k} \).

\[ 9.4 \]

\[
\psi_1 = A_1 e^{\kappa x}, \quad x < -L \\
\psi_{II} = A_2 \cos(kx) + B_2 \sin(kx), \quad -L \leq x \leq L \\
\psi_{III} = B_3 e^{-\kappa x}, \quad x > L
\]

\[
\kappa = \sqrt{-\frac{2m_0E}{\hbar^2}}, \quad k = \sqrt{\frac{2m_0(E + V_0)}{\hbar^2}}.
\]

The requirements that the solutions be continuous and differentiable at \( x = \pm L \) yield:

\[
k = k \tan(kL) \to E(L, V_0) \quad (\text{graphically or numerically}),
\]

and the two types of solutions:

\[
A_1 = B_3, \quad B_2 = 0 \quad \text{symmetric solution}; \\
A_1 = -B_3, \quad B_2 \neq 0 \quad \text{antisymmetric solution}.
\]

\[ 9.5 \]

\[
a = \frac{1}{1 - \frac{m_0\beta}{\hbar^2k}} - 1 \\
b = \frac{1}{1 - \frac{m_0\beta}{\hbar^2k}}, \quad \text{with} \quad k = \sqrt{\frac{2m_0E}{\hbar^2}}.
\]

The meanings of \( a \) and \( b \):

\[
|a|^2 = \text{reflection coefficient}, \\
|b|^2 = \text{transmission coefficient}.
\]

\[
|a|^2 = \frac{1}{\frac{m_0^2\beta^2}{\hbar^4k^2} + 1}
\]

\[ 9.6 \] The wall is located at the position \( x = 0 \). Since \( \psi(x) \) must be continuous at \( x = 0 \), we find \( \psi(0) = 0 \).

For the wavefunctions, the result is:

\[
\psi(x) = \begin{cases} 
A \sin(kx), & x \leq 0 \\
0, & x \geq 0
\end{cases}
\]

with \( k = \sqrt{\frac{2m_0E}{\hbar^2}} \),

where \( E \) is the energy of the particle.

\[ 9.7 \]

\[
\langle x \rangle = \int \psi^*(x, t) x \psi(x, t) \, dx = 0
\]

\[
\langle p \rangle = \int \psi^*(x, t) \left(-i\hbar \frac{d}{dx}\right) \psi(x, t) \, dx = 0
\]

\[
\langle E_{\text{kin}} \rangle = \frac{\langle p^2 \rangle}{2m_0} = \int \psi^*(x, t) \left(-i\hbar \frac{d}{dx}\right)^2 \psi(x, t) \, dx
\]

\[
= \frac{\hbar^2(\Delta k)^2}{4m_0}
\]

\[
\langle x^2 \rangle = \int \psi^*(x, t) x^2 \psi(x, t) \, dx
\]

\[
= \frac{1}{2(\Delta k)^2} + \frac{\hbar(\Delta k)^2}{2m_0} \langle \xi^2 \rangle
\]

\( \langle x^2 \rangle \) illustrates directly the spreading out of the wave packet.

\[ 9.8 \]

\[
\bar{E} = \frac{1}{\sqrt{\pi(\Delta k)^2}} \int_{-\infty}^{+\infty} \frac{dk}{2m_0} \frac{\hbar^2k^2}{2m_0} \exp\{-k^2/(\Delta k)^2\}
\]

\[ 9.9 \]

\[
[i_x, i_y] = -\hbar^2 \left( \begin{array}{c}
\frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \\
\frac{\partial}{\partial x} - x \frac{\partial}{\partial z}
\end{array} \right)
\]

\[
- \left( \begin{array}{c}
\frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \\
\frac{\partial}{\partial z} - z \frac{\partial}{\partial y}
\end{array} \right)
\]

\[
= -\hbar^2 \left( \begin{array}{c}
\frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \\
\frac{\partial}{\partial z} - z \frac{\partial}{\partial y}
\end{array} \right) = i\hbar i_z
\]
\[ \{\hat{y}, \hat{z}\} \text{ and } \{\hat{z}, \hat{x}\} \text{ analogously.} \]

\[ [\hat{r}^2, \hat{r}] = [\hat{r}^2, \hat{r}_y] + [\hat{r}^2, \hat{r}_x] + [\hat{r}^2, \hat{r}_z] \]

\[ = \hat{r}_y[\hat{r}_x, \hat{r}] + [\hat{r}_x, \hat{r}] \hat{r}_y + \hat{r}_z[\hat{r}_x, \hat{r}] \hat{r}_y + [\hat{r}_x, \hat{r}_z] \hat{r}_y \]

\[ = -i \hbar \hat{r}_y \hat{r}_z - i \hbar \hat{r}_z \hat{r}_y + i \hbar \hat{r}_x \hat{r}_y + i \hbar \hat{r}_y \hat{r}_x \]

\[ [\hat{r}^2, \hat{r}_y] \text{ and } [\hat{r}^2, \hat{r}_z] \text{ analogously.} \]

9.10 \[ [\hat{r}_x, x] = 0 \text{ and } [\hat{r}_x, V(r)] = 0 \]

9.11 a) \((ab)^* = a^* b^* \) with \( x \) and \( dx \) real.

b) \[ \int_{-\infty}^{\infty} \psi_2^* \frac{\hbar}{i} \frac{\partial}{\partial x} \psi_2 dx = \left[ \frac{\hbar}{i} \psi_1^* \psi_2 \right]_{-\infty}^{\infty} \]

\[ - \int_{-\infty}^{\infty} \psi_2^* \frac{\partial}{\partial x} \psi_2 dx \]

\[ = \left( \int_{-\infty}^{\infty} \psi_2^* \frac{\partial}{\partial x} \psi_2 \right)^* \]

c) \[ \int_{-\infty}^{\infty} \psi_2^* \frac{\partial^2}{\partial x^2} \psi_2 dx = \left[ \psi_1^* \frac{\partial}{\partial x} \psi_2 \right]_{-\infty}^{\infty} \]

\[ - \int_{-\infty}^{\infty} \left( \frac{\partial}{\partial x} \psi_1^* \right) \left( \frac{\partial}{\partial x} \psi_2 \right) dx \]

\[ = \left[ - \frac{\partial}{\partial x} (\psi_1^* \psi_2) \right]_{-\infty}^{\infty} \]

\[ + \int_{-\infty}^{\infty} \psi_2 \frac{\partial^2}{\partial x^2} \psi_1^* dx \]

\[ = \left( \int_{-\infty}^{\infty} \psi_2 \frac{\partial^2}{\partial x^2} \psi_1^* dx \right)^* \]

9.12 \[ \frac{d}{dt} \langle \psi \rangle = \int \psi^* x \psi dx + \int \psi^* x \psi dx \]

with \( \dot{\psi} = \frac{1}{i\hbar} \left( -\frac{\hbar^2}{2m_0} \frac{d^2}{dx^2} + V \right) \psi \)

Substituting:

\[ \frac{i\hbar}{2m_0} \int dx \left( \psi^* x \frac{d^2}{dx^2} \psi - \left( \frac{d^2}{dx^2} \psi^* \right) x \psi \right) \]

partial integration:

\[ \frac{i\hbar}{2m_0} \int dx \left( \psi \frac{d}{dx} \psi^* - \left( \frac{d}{dx} \psi \right) \psi^* \right) \]

partial integration:

\[ \frac{1}{m_0} \int dx \psi^* \psi' = \frac{1}{m_0} \langle \psi^* \psi' \rangle \]

\[ \frac{d}{dt} \langle \psi \rangle = -i\hbar \int \dot{\psi}^* \frac{d}{dx} \psi dx - i\hbar \int \psi^* \frac{d}{dx} \psi dx \]

Substitute in the Schrödinger equation and integrate by parts two times:

\[ \frac{d}{dt} \langle \psi \rangle = \int dx \psi^* \left( -\frac{d}{dx} V \right) \psi = -\int \frac{dV}{dx} dx \]

In three dimensions, we have for each component \( i \):

\[ m_0 \frac{d}{dt} \langle \psi \rangle_{i} = \langle \psi_{i} \rangle , \quad \frac{d}{dt} \langle \psi \rangle_{i} = -\int \frac{dV}{dx_{i}} dx_{i} \]

9.13 The potential is \( V(x) = \frac{1}{2} m_0 \omega^2 (x-x_0)^2 - \epsilon_0 \)

with \( x_0 = \frac{k_0}{k} \) and \( \epsilon_0 = \frac{1}{2} \frac{k_0^2}{k} \).

The time-independent Schrödinger equation is then

\[ -\frac{\hbar^2}{2m_0} \frac{d^2}{dx^2} + \frac{1}{2} m_0 \omega^2 (x-x_0)^2 - \epsilon_0 \]

\[ = E \psi(x) \]

Employing the transformations \( y = x - x_0 \) and \( \hat{E} = E + \epsilon_0 \), we obtain the well-known form

\[ -\frac{\hbar^2}{2m_0} \frac{d^2}{dy^2} + \frac{1}{2} m_0 \omega^2 y^2 \]

\[ = \hat{E} \psi(y) \]

The solutions of this equation are transformed back, and we find for the eigenfunctions:

\[ \psi_{n}(x) = \left( \frac{m_0 \omega}{\hbar} \right)^{1/4} \exp \left[ -\frac{m_0 \omega}{2 \hbar} (x-x_0)^2 \right] \]
Solutions to the Problems

9.14 \[ b = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial \xi} + \xi \right) ; \quad b^+ = \frac{1}{\sqrt{2}} \left( -\frac{\partial}{\partial \xi} + \xi \right) ; \]

\[ \xi = \sqrt{\frac{m_0 \omega}{\hbar}} x \]

\[ [b, b^+] = \frac{1}{2} \left\{ \left( \frac{\partial}{\partial \xi} + \xi \right) \left( -\frac{\partial}{\partial \xi} + \xi \right) \right. \]

\[ \left. - \left( -\frac{\partial}{\partial \xi} + \xi \right) \left( \frac{\partial}{\partial \xi} + \xi \right) \right\} \]

\[ = \frac{1}{2} \left\{ -\frac{\partial^2}{\partial \xi^2} - \xi \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \xi} \xi + \frac{\partial^2}{\partial \xi^2} \right. \]

\[ - \xi \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \xi} \xi + \xi^2 - \xi^2 \}

\[ = -\xi \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \xi} \xi = -\xi \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial \xi} + 1 = 1. \]

9.15 \[ \psi(t) = \exp \left( -i \frac{\omega}{2} t \right) [\psi_0 + \psi_1 \exp(-i \omega t)] \]

Since \( \psi_0 \) and \( \psi_1 \) are real, we obtain

\[ |\psi(t)|^2 = \psi_0^2 + \psi_1^2 + 2 \psi_0 \psi_1 \cos(\omega t) . \]

If one chooses e.g. \( t_n = 2 \pi n/\omega \), with \( n = 0, 1, 2, \ldots \), the result is

\[ |\psi(t_n)|^2 = (\psi_0 + \psi_1)^2 , \quad \text{and} \]

\[ |\psi(t_n + \pi/\omega)|^2 = (\psi_0 - \psi_1)^2 . \]

The wavefunction thus oscillates periodically between the two forms.

9.16 a) \[ \int_{-\infty}^{\infty} \left( \frac{\partial u}{\partial \xi} \right) v(\xi) d\xi = [u(\xi) v(\xi)]_{-\infty}^{\infty} \]

\[ - \int_{-\infty}^{\infty} u(\xi) \left( \frac{\partial v}{\partial \xi} \right) d\xi . \]

For functions which vanish at infinity, then,

\[ \int_{-\infty}^{\infty} \left( \frac{\partial u}{\partial \xi} \right) v(\xi) d\xi = \int_{-\infty}^{\infty} u(\xi) \left( -\frac{\partial v}{\partial \xi} \right) d\xi . \]

b) \[ \int (b^+ \phi_n) (b^+ \phi_n) d\xi = \int \phi_n^* b b^+ \phi_n d\xi = \int \phi_n^* (1 + b^+ b) \phi_n d\xi = (1 + n) \int \phi_n^* \phi_n d\xi \]

with \( \frac{1}{b} = \int \phi_n^* \phi_n d\xi \)

d) \[ \phi_n = \frac{1}{\sqrt{n!}} (b^+)^n \phi_0 \quad \text{follows immediately from (c).} \]

e) With \( \phi_{n+1} = \frac{1}{\sqrt{n+1}} b^+ \phi_n \quad \text{we have} \]

\[ b^+ \phi_n = \sqrt{n+1} \phi_{n+1} ; \]

from \( \phi_n = \frac{1}{\sqrt{n}} b^+ \phi_{n-1} \quad \text{we have} \]

\[ b \phi_n = \frac{1}{\sqrt{n}} b b^+ \phi_{n-1} = \frac{1}{\sqrt{n}} (1 + b^+ b) \phi_{n-1} \]

\[ = \frac{1 + n - 1}{\sqrt{n}} \phi_{n-1} ; \]

and therefore:

\[ b \phi_n = \sqrt{n} \phi_{n-1} . \]

f) Assuming

\[ K_n = [b, (b^+)^n] = n (b^+)^{n-1} \]

is valid for a particular \( n \), then it follows that
\[ K_{n+1} = [b, (b^+)^{n+1}] = b(b^+)^n b^+ - (b^+)^{n+1} b \]
\[ = ((b^+)^n b + K_n) b^+ - (b^+)^{n+1} b \]
\[ = (b^+)^n b b^+ + n(b^+)^n b^+ - (b^+)^n b + n(b^+)^n b^+ \]
\[ = (b^+)^n (b b^+ + 1) + n(b^+)^n \]
\[ = (n+1)(b^+)^n . \]

Thus, the assumption is also valid for \( n+1 \). Since it is valid for \( n = 1 \), it is valid for all \( n \)!

Analogously:
\[ Q_n = [b^+, b^n] = -n b^{n-1} \]

is supposed to be valid for a particular \( n \). Then we also have
\[ Q_{n+1} = b^+ b^{n+1} - b^{n+1} b^+ \]
\[ = (b^n b^+ + Q_n) b - b^{n+1} b^+ \]
\[ = b^n b^+ b - n b^n - b^{n+1} b^+ \]
\[ = b^n (b b^+ - 1) - n b^n - b^{n+1} b^+ \]
\[ = -(n+1) b^n . \]

Thus, the assumption is also valid for \( n+1 \). Since it is valid for \( n = 1 \), it is also valid for all \( n \)!

9.17 Momentum \( p \sim (b - b^+)/i \), position \( x \sim (b + b^+) \) with
\[ \int \phi_n^* b \phi_n d\xi = \sqrt{n} \int \phi_n^{*1} \phi_{n-1} d\xi = 0 \quad \text{and} \]
\[ \int \phi_n^* b^+ \phi_n d\xi = (1/\sqrt{n+1})^{-1} \int \phi_n^{*1} \phi_{n+1} d\xi = 0 \]
it follows that
\[ \int \phi_n^* p \phi_n d\xi = \int \phi_n^* x \phi_n d\xi = 0 . \]

Kinetic energy:
\[ \frac{1}{2} p^2 \sim -\frac{1}{4} (b - b^+)^2 \]
\[ = -\frac{1}{4} (b b^+ b^+ b^+ - b b^+ b^+ b^+ + b^+ b) \]
\[ = -\frac{1}{4} (b^+ b^+ b b^+ b^+ - 2 b^+ b b^+ b - 1) . \]
\[ \int \phi_n^* \frac{p^2}{2} \phi_n d\xi \quad \text{is then} \quad \int \phi_n^* \left( \frac{1}{2} b^+ b + \frac{1}{4} \right) \phi_n d\xi \]
\[ = \frac{1}{2} (n + \frac{1}{2}) . \]

Analogously for the potential energy:
\[ \frac{1}{2} x^2 \sim \frac{1}{4} (b + b^+)^2 = \frac{1}{4} (b^+ b^+ + b b + 2 b^+ b + 1) \]
\[ = \int \phi_n^* x^2 \phi_n d\xi \rightarrow \frac{1}{2} \left( n + \frac{1}{2} \right) . \]

9.18 To be shown:
\[ \int \phi_n^* \phi_m d\xi = \delta_{n,m} . \]

\( \phi_n \) and \( \phi_m \) are solutions of the Schrödinger equations
\[ b^+ b \phi_n = n \phi_n \quad \text{and} \quad b^+ b \phi_m = m \phi_m . \]

Multiplying by \( \phi_m \) or \( \phi_n \), we obtain
\[ \int \phi_n^* b^+ b \phi_n d\xi = n \int \phi_n^* \phi_n d\xi \quad \text{and} \quad (1) \]
\[ \int (b^+ b \phi_m)^* \phi_n d\xi = m \int \phi_m^* \phi_n d\xi \quad \text{and} \quad (2) \]

Taking the difference between (1) and (2), we find
\[ 0 = (n - m) \int \phi_n^* \phi_n d\xi , \]
i.e. the eigenfunctions are orthogonal and — with Problem 9.16c — are normalised.

9.19 a) as in Problem 9.16e
b) as in Problem 9.18
c), d) as in Problem 9.17.

10.1 a) \( \bar{E}_{\text{kin}} = \frac{1}{2} \frac{m_0 e^4}{(4 \pi \varepsilon_0)^2 \hbar^2} \) \( \bar{E}_{\text{pot}} = -\frac{m_0 e^4}{(4 \pi \varepsilon_0)^2 \hbar^2} \)
b) \( \bar{E}_{\text{kin}} = \frac{1}{8} \frac{m_0 e^4}{(4 \pi \varepsilon_0)^2 \hbar^2} \) \( \bar{E}_{\text{pot}} = -\frac{1}{4} \frac{m_0 e^4}{(4 \pi \varepsilon_0)^2 \hbar^2} \)

10.2 a) \( D = (0, 0, 0) \)
b) 1. \( D = (0, 0, 0) \)
2. \( D = d / \sqrt{3} (0, 0, 1) \)
3. \( D = \frac{1}{3} \sqrt{3/2} d (-1, \pm i, 0) \)
\[ d = \frac{256 \sqrt{6}}{81} \frac{h^2}{m_0 e^2 \cdot 4 \pi \varepsilon_0} \]
10.3  \( \kappa_1 = [0.529 \cdot 10^{-8} \text{ cm}]^{-1} \quad E_1 = -13.55 \text{ eV} \)
\( \kappa_2 = [1.058 \cdot 10^{-8} \text{ cm}]^{-1} \quad E_2 = -3.39 \text{ eV} \)
\( \kappa_3 = [1.587 \cdot 10^{-8} \text{ cm}]^{-1} \quad E_3 = -1.51 \text{ eV} \)

10.4  a) \( N = (2/\pi)^{3/4}(r_0)^{-3/2} \)
\[ r_{0 \min} = 6\pi^{3/2} \varepsilon_0 \hbar^2/(\sqrt{2} e^2 m_0) \]
\[ \tilde{E}_{\min} = -e^4 m_0/(12 \hbar^2 \pi^3 \varepsilon_0^2) = \frac{8}{3 \pi} E_0 \]

b) \( N = (\pi r_0^3)^{-1/2} \)
\[ r_{0 \min} = 4\pi \varepsilon_0 \hbar^2/(m_0 e^2) \]
\[ \tilde{E}_{\min} = -e^4 m_0/(32 \hbar^2 \pi^2 \varepsilon_0^2) = E_0 \]
\( E_0 \): exact ground-state energy.

10.5  Differential equation for \( g(x) \):
\[ \frac{d^2}{dx^2} g(x) + 2 \left( \frac{\sigma}{x} - |\varepsilon| \right) \frac{d}{dx} g(x) \]
\[ + \frac{g(x)}{x} (\hat{c}_1 - 2 \sigma |\varepsilon|) = 0 \]

Power series expansion for \( g(x) \) as trial solution:
\[ g(x) = \sum_{n=0}^{\infty} a_n x^n \]

Recursion formula for the expansion coefficients \( a_n \):
\[ a_{n+1} = a_n \left( 2n \sqrt{|\varepsilon|} - \hat{c}_1 + 2 \sigma \sqrt{|\varepsilon|} \right)/[(n+1) (n+2 \sigma)] \]

Energy eigenvalues:
\[ E_n = -m_0 c_1^2 \left[ n + \frac{1}{2} + \frac{1}{2} \sqrt{1 + 8 m_0 c_2^2 / \hbar^2} \right]^{-2} \]

11.1

<table>
<thead>
<tr>
<th>( E_{n,l} ) [eV]</th>
<th>( l = 0 )</th>
<th>( l = 1 )</th>
<th>( l = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Li:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( n = 2 )</td>
<td>-5.31</td>
<td>-3.54</td>
<td></td>
</tr>
<tr>
<td>( n = 3 )</td>
<td>-2.01</td>
<td>-1.55</td>
<td>-1.51</td>
</tr>
<tr>
<td>( n = 4 )</td>
<td>-1.05</td>
<td>-0.87</td>
<td>-0.85</td>
</tr>
<tr>
<td>Na:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( n = 3 )</td>
<td>-5.12</td>
<td>-3.03</td>
<td>-1.52</td>
</tr>
<tr>
<td>( n = 4 )</td>
<td>-1.97</td>
<td>-1.40</td>
<td>-0.85</td>
</tr>
<tr>
<td>( n = 5 )</td>
<td>-1.03</td>
<td>-0.80</td>
<td>-0.55</td>
</tr>
</tbody>
</table>

11.2  \( E_{n,l} = -\frac{R_{\text{alkali}} \hbar c}{(n - \Delta(l))^2} \);  
\( \Delta(0) = 0.41; \quad \Delta(1) = 0.04 \)

11.3

<table>
<thead>
<tr>
<th>Transition</th>
<th>( \tilde{\nu} ) [cm(^{-1})]</th>
<th>( \lambda ) [nm]</th>
<th>Region</th>
</tr>
</thead>
<tbody>
<tr>
<td>3p ( \rightarrow ) 2s</td>
<td>31000</td>
<td>323</td>
<td>UV</td>
</tr>
<tr>
<td>3p ( \rightarrow ) 3s</td>
<td>3800</td>
<td>2619</td>
<td>far IR</td>
</tr>
<tr>
<td>3s ( \rightarrow ) 2p</td>
<td>12200</td>
<td>820</td>
<td>IR</td>
</tr>
<tr>
<td>2p ( \rightarrow ) 2s</td>
<td>14900</td>
<td>671</td>
<td>visible</td>
</tr>
</tbody>
</table>

11.3 | 3\(^2\)D | 3\(^2\)P
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Principal quantum number</td>
<td>( n )</td>
<td>3</td>
</tr>
<tr>
<td>Spin quantum number</td>
<td>( s )</td>
<td>1/2</td>
</tr>
<tr>
<td>Angular momentum quantum number</td>
<td>( l )</td>
<td>2</td>
</tr>
</tbody>
</table>

Three lines result from this transition:

\( 3\(^2\)D_{5/2} \rightarrow 3\(^2\)P_{3/2} \)
\( 3\(^2\)D_{3/2} \rightarrow 3\(^2\)P_{3/2} \)
\( 3\(^2\)D_{3/2} \rightarrow 3\(^2\)P_{1/2} \).

The new index represents the total angular momentum quantum number \( J = L \pm 1/2 \).

12.1  Electron:
\[ \omega_L = 2\pi \nu_L = \frac{2\mu_B B}{\hbar} = 3.52 \cdot 10^6 \text{ s}^{-1} \]
Proton:
\[ \omega_L = 2\pi \nu_L = \frac{5.585 \mu_B B}{\hbar} = 5345 \text{ s}^{-1} \]
with \( \mu_N = \mu_B/1836 \).

12.2  Result:
\[ \mu_z = \frac{mdv^2}{2\partial B_z/\partial l_1 (l_2 + l_1/2)} = 9.32 \cdot 10^{-24} \text{ Am}^2 \]
The nuclear spin contributes to the deflection by an amount which is 3 orders of magnitude smaller and can thus be neglected.

12.3  With \( l = n \hbar \) (cf. Bohr model) we have from Sect. 12.8
\[ B_l = \frac{e\mu_0}{4\pi r^3 m_0} l = 12.54 \text{ T} \]
12.4 In general (compare Sect. 12.2):
\[ \mu_l = g_l \mu_B \sqrt{l(l+1)} ; \quad \mu_{l,z} = \mu_B m_l . \]

Electron: \( \mu_B = \frac{e h}{2 m_0} = 9.274 \cdot 10^{-24} \text{ Am}^2 \)

Muon: \( \mu_B(\text{muon}) = \mu_B/207 = 4.480 \cdot 10^{-26} \text{ Am}^2 \).

In the case of positronium, due to the opposite charges and equal masses, no current flows and the magnetic moment is zero.

12.5 \( V_{l,s} = \frac{V_0}{n^2 l(l+\frac{1}{2})(l+1)} \cdot (j(j+1) - l(l+1) - s(s+1)) \)

with \( V_0 = \frac{Z^4 e^2 \mu_0 h^2}{16 \pi m_0^2 a_0^3} = 5.81 \cdot 10^{-23} \text{ J} = 3.63 \cdot 10^{-4} \text{ eV} \)

12.6 The energy difference \( \Delta E \) between the states:
\[ \Delta E = \frac{\hbar c}{\lambda_0} \left( 1 - \frac{1}{\lambda_0 + \Delta \lambda} \right) = 1.101 \cdot 10^{-20} \text{ J} = 0.0687 \text{ eV} \]

with \( \lambda_0 = 852.1 \text{ nm} \), \( \Delta \lambda = 422 \text{ nm} \).

The fine-structure interaction energy (cf. Sect. 12.8):
\[ V_{l,s} = -\mu_s \cdot B_i = g_s \mu_B \sqrt{s(s+1)} B_i \cos (\chi(s, B_i)) \]
\[ = \frac{a}{2} (j(j+1) - l(l+1) - s(s+1)) \]

(For the calculation of \( \cos (\chi(s, B_i)) \) see Sect. 12.8.)

\[ \to \Delta E = V_{l,s} (j = 3/2) - V_{l,s} (j = 1/2) \]
\[ \to B_i = 559.5 \text{ T}; \quad a = 7.339 \cdot 10^{-21} \text{ J} = 0.0458 \text{ eV} . \]

12.7 1) Energy states of the hydrogen atom as described by the Dirac theory (cf. Sect. 12.10):
\[ E_{n,j} = E_n + \Delta E_{n,j} = E_n + \frac{E_n \alpha^2}{n} \left( \frac{1}{j+1/2} - \frac{3}{4n} \right) \]
with \( E_n = -\hbar c R_H/n^2 \).

2) The Lamb shift lifts the degeneracy of the states with the same value of \( n \), so that states with smaller \( l \) values have slightly higher energies.

3) Selection rules for the possible transitions:
\( \Delta l = \pm 1; \quad \Delta j = 0, \pm 1 \).

A total of 18 lines occur; see Fig. 12.20 (taking the lifting of the \( J \) degeneracy into account).

12.8 a) \[ -\frac{E_{FS} n^3}{\hbar c R Z^4 \alpha^2} \]
\[ = \left( \frac{1}{j+1/2} - \frac{3}{4n} \right) > \frac{1}{j_{\max} + 1/2} - \frac{3}{4n} \]
\[ = \frac{1}{4n} > 0 \]

with \( j_{\max} = n - 1/2 \). Then, for all \( n \) and \( j \), we have \( E_{FS} < 0 \).

b) In hydrogen-like atoms, the energies of the electronic states depend only on \( n \) and \( j \). For a given \( n \), there are just \( n \) different \( j \) values: 1/2, 3/2, . . . , \( n - 1/2 \). Thus, the terms for \( n = 3 \) and \( n = 4 \) are split into 3 or 4 levels, respectively.

c) \[ \Delta \tilde{v} = \frac{\Delta E_{FS}}{\hbar c} = -\frac{\alpha^2 R Z^4}{n^3} \left( \frac{1}{j+1/2} - \frac{3}{4n} \right) \]

\[ n \quad 3 \quad 3 \quad 3 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4 \]
\[ j \quad 1/2 \quad 3/2 \quad 5/2 \quad 1/2 \quad 3/2 \quad 5/2 \quad 7/2 \]
\[ -\Delta \tilde{v} \quad 0.162 \quad 0.054 \quad 0.018 \quad 0.074 \quad 0.029 \quad 0.013 \quad 0.006 \quad [\text{cm}^{-1}] \]

(for \( Z = 1 \)).

d) Selection rules: \( \Delta l = \pm 1; \quad \Delta j = 0, \pm 1 \).

The following table shows the possible transitions and the deviations in the positions of the lines as a result of the fine-structure interaction.
### 12.9 Fine structure (spin-orbit) interaction:

\[ V_{L, S} = \frac{A}{2} [J(J + 1) - L(L + 1) - S(S + 1)] \]

The fine-structure constant \( A \) is a constant for a particular multiplet \( (L, S = \text{constant}) \).

Selection rules: \( \Delta L = 0 \), \( \pm 1 \) \( \Delta J = 0 \), \( \pm 1 \)

(for the case of \( \Delta L = 0 \) see Sect. 17.3.2)

(see figure on the right)

### 12.10

a) A magnetic dipole always produces a magnetic field \( B(r) \) in its neighbourhood:

\[
B(r) = -\frac{\mu_0}{4\pi} \text{grad} \left( \frac{\mu \cdot r}{r^3} \right) = \frac{\mu_0}{4\pi} \left( \frac{3(\mu \cdot r) r - \mu r^2}{r^5} \right).
\]

Thus, for the interaction between two dipoles,

\[
E = -\mu_2 \cdot B_1(r) = \frac{\mu_0}{4\pi} \left[ \frac{\mu_1 \cdot \mu_2}{r^3} - 3 \frac{\mu_1 \cdot r(\mu_2 \cdot r)}{r^5} \right]
\]

\( E \) is always zero when \( \mu_2 \perp B_1(r) \).

b) If \( \mu_1 \) and \( \mu_2 \) are parallel and \( \theta \) is the angle between \( \mu_1 \) or \( \mu_2 \) and \( r \), we have

\[
E = -\frac{\mu_0}{4\pi} \frac{\mu_1 \mu_2}{r^3} (1 - 3 \cos^2 \theta).
\]

Extreme values for fixed \( |r| = r \):

\[
\theta = 0, \pi: \quad E = -\frac{\mu_0}{2\pi} \frac{\mu_1 \mu_2}{r^3},
\]

\[
\theta = \pi/2: \quad E = \frac{\mu_0}{4\pi} \frac{\mu_1 \mu_2}{r^3}.
\]

c) Electron-electron:

\[
\mu_s = g_s \mu_B \sqrt{s(s + 1)}; \quad s = 1/2; \quad \mu_B = \frac{e\hbar}{2m_0}; \quad g_s = 2
\]

Diagram for Solution 12.9
13.1 \( \nu = \frac{2\mu_B B_z}{\hbar} = 2.8 \cdot 10^9 \text{ Hz} = 2.8 \text{ GHz} \)

13.2 \( ^4D_{1/2} \): \( L = 2, S = 3/2, J = 1/2 \)

The resultant vector \( \mu_J \) for the magnetic moment is perpendicular to \( J \) and precesses around it. Only the component \( (\mu_J)_J \) which is parallel to \( J \) is measurable; however, it does not occur here (cf. Fig. 13.9).

Therefore, the \( ^4D_{1/2} \) term does not split in a magnetic field.

### Solutions to the Problems

| Angle \( \theta \) | Energy \( E \) | Magnetic field \( |B_i(r)| \) |
|-------------------|-------------|------------------|
| 0, \( \pi \)      | \(-6.451 \cdot 10^{-24} \text{ J}\) | 0.402 T |
| \( \pi/2 \)       | \(3.226 \cdot 10^{-24} \text{ J}\) | 0.201 T |

Proton-proton:

\[ \mu_p = g_p \mu_N \sqrt{I(I+1)}; \quad I = 1/2; \quad \mu_N = \frac{\mu_B}{1836.15}; \]

\[ g_p = 5.585 \]

| Angle \( \theta \) | Energy \( E \) | Magnetic field \( |B_i(r)| \) |
|-------------------|-------------|------------------|
| 0, \( \pi \)      | \(-1.492 \cdot 10^{-29} \text{ J}\) | 6.11 \cdot 10^{-4} \text{ T} |
| \( \pi/2 \)       | \(7.46 \cdot 10^{-30} \text{ J}\) | 3.05 \cdot 10^{-4} \text{ T} |

13.3 \( \phi = \frac{\pi}{3} (L, J) = 50.77^\circ \)

13.4 a) The energy difference between the \( ^2P_{3/2} \) and \( ^2P_{1/2} \) states without a magnetic field is

\[ \Delta E = \frac{\hbar c}{\lambda_1} - \frac{\hbar c}{\lambda_2}. \]

Additional energy in a magnetic field:

\[ ^2P_{1/2}: \quad l = 1, j = 1/2 \quad g_j = 2/3 \]

\[ V_{mj}(\frac{1}{2}, 1) = \frac{3}{2} \mu_B B_0 m_j \]

\[ ^2P_{3/2}: \quad l = 1, j = 3/2 \quad g_j = 4/3 \]

\[ V_{mj}(\frac{3}{2}, 1) = \frac{4}{3} \mu_B B_0 m_j \]

(cf. Sect. 13.3.4 and 5).

Condition:

\[ V_{1/2}(\frac{1}{2}, 1) - V_{-3/2}(\frac{3}{2}, 1) = \Delta E \]

\[ B_0 = \frac{3 \hbar c}{7 \mu_B} \left( \frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right) = 15.77 \text{ T} . \]

b) \( D_1 \) line: \( \Delta \nu = \frac{8 \mu_B B}{3 \hbar} = 3.73 \cdot 10^{10} \text{ Hz} \)

\[ \Delta \nu = \frac{10 \mu_B B}{3 \hbar} = 4.67 \cdot 10^{10} \text{ Hz} . \]

13.5 The extra term in the energy for the anomalous Zeeman effect:

\[ \Delta E_B = g_j \mu_B B m_j; \]

\[ g_j = 1 + \frac{j(j+1) + s(s+1) - l(l+1)}{2j(j+1)} \]

Landé factors:

\[ g_j(^2D_{5/2}) = 6/5; \quad g_j(^2D_{3/2}) = 4/5; \quad g_j(^2P_{3/2}) = 4/3; \]

\[ g_j(^2P_{1/2}) = 2/3. \]

Selection rules: \( \Delta l = \pm 1, \Delta j = 0, \pm 1, \Delta m_j = 0, \pm 1. \)

1) \( ^2D_{5/2} \leftrightarrow ^2P_{3/2} \) transition:

\[ \frac{\Delta (h \nu)}{\mu_B B} = g_j(^2D_{5/2}) \cdot m_j(^2D_{5/2}) - g_j(^2P_{3/2}) \cdot m_j(^2P_{3/2}) \]
Solutions to the Problems

\[ r_{ij} = \frac{3}{2} \]
\[ \omega_{js} = \frac{1}{2} \]
\[ \omega_{js} = -\frac{1}{2} \]
\[ \mu_{j} = -\frac{3}{2} \]

\[ \begin{array}{c|cccccc}
\hline
2D_{3/2} & m_j = 3/2 & m_j = 1/2 & m_j = -1/2 & m_j = -3/2 & m_j = 5/2 \\
\hline
m_j = 3/2 & 1 & 1/5 & 7/5 & & \\
m_j = 1/2 & 17/15 & 1 & 15 & -19/15 & \\
m_j = -1/2 & 19/15 & 1/15 & 17/15 & & \\
m_j = -3/2 & 7/5 & 1/5 & 1 & & \\
\hline
\end{array} \]

2) \( ^2D_{3/2} \leftrightarrow ^2P_{3/2} \)-transition:
\[
\frac{\Delta (h \nu)}{\mu_B B} = g_j (^2D_{3/2}) \cdot m_j (^2D_{3/2})
\]
\[
- g_j (^2P_{3/2}) \cdot m_j (^2P_{3/2})
\]

\[ \begin{array}{c|cccc}
\hline
2D_{3/2} & m_j = 3/2 & m_j = 1/2 & m_j = -1/2 & m_j = -3/2 \\
\hline
m_j = 3/2 & -4/5 & -8/5 & & \\
m_j = 1/2 & 8/15 & -4/15 & 16/15 & \\
m_j = -1/2 & 16/15 & 4/15 & -8/15 & \\
m_j = -3/2 & 8/5 & 4/5 & & \\
\hline
\end{array} \]

3) \( ^2D_{3/2} \leftrightarrow ^2P_{1/2} \)-transition:
\[
\frac{\Delta (h \nu)}{\mu_B B} = g_j (^2D_{3/2}) \cdot m_j (^2D_{3/2})
\]
\[
- g_j (^2P_{1/2}) \cdot m_j (^2P_{1/2})
\]

\[ \begin{array}{c|cccc}
\hline
2D_{3/2} & m_j = 3/2 & m_j = 1/2 & m_j = -1/2 & m_j = -3/2 \\
\hline
m_j = 1/2 & 13/15 & 1/15 & -11/15 & \\
m_j = -1/2 & 11/15 & -1/15 & 13/15 & \\
\hline
\end{array} \]

13.6 a) No magnetic field: Each line in the Balmer series, neglecting the Lamb shift, consists of five individual lines.

b) Weak magnetic field: Anomalous Zeeman effect.

<table>
<thead>
<tr>
<th>Transition</th>
<th>F - S Energy</th>
<th>Number of Zeeman lines</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n^2D_{3/2} \rightarrow 2^2P_{3/2} )</td>
<td>0.01254</td>
<td>12</td>
</tr>
<tr>
<td>( n^2D_{3/2} \rightarrow 2^2P_{3/2} )</td>
<td>0.06366</td>
<td>10</td>
</tr>
<tr>
<td>( n^2D_{3/2} \rightarrow 2^2P_{3/2} )</td>
<td>0.06887</td>
<td>6</td>
</tr>
<tr>
<td>( n^2P_{3/2} \rightarrow 2^2S_{1/2} )</td>
<td>0.06887</td>
<td>6</td>
</tr>
<tr>
<td>( n^2P_{3/2} \rightarrow 2^2S_{1/2} )</td>
<td>0.05035</td>
<td>4</td>
</tr>
<tr>
<td>( n^2S_{1/2} \rightarrow 2^2P_{3/2} )</td>
<td>0.05035</td>
<td>4</td>
</tr>
<tr>
<td>( n^2S_{1/2} \rightarrow 2^2P_{3/2} )</td>
<td>-0.01215</td>
<td>6</td>
</tr>
</tbody>
</table>

(cf. also Problem 13.5).

3) \( ^2D_{3/2} \rightarrow ^2P_{3/2} \)-transition:
\[
\Delta E \approx h c \cdot 0.1 \text{ cm}^{-1} \approx \mu_B B_{\lim}
\]

Estimate of the transition from the Zeeman to the Paschen-Back region:
\[
\Delta E_{F - S} = h c \cdot 0.1 \text{ cm}^{-1} \approx \mu_B B_{\lim}
\]
thus, \( B_{\lim} \approx 0.2 \text{ T} \).

13.7 a) According to Problem 13.6, the transition from the Zeeman to the Paschen-Back region occurs around \( B = 0.2 \text{ T} \), i.e. here we are in the region of the Paschen-Back effect.

b) \( \Delta E_{m_j, m_s} = -\mu_{l, z} B_0 - \mu_{s, z} B_0 = \mu_B B_0 (m_l + 2m_s) \).

Note that \( \Delta E_{m_j, m_s} \) is the deviation of the energy from that of the same state without fine-structure splitting.
Selection rules: $\Delta m_l = 0, \pm 1; \Delta m_s = 0$. 

<table>
<thead>
<tr>
<th>without B-field coupling</th>
<th>only with $I$ coupling with $s$ and $I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_l$</td>
<td>$m_s$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>0</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>-1</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>0</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>-1</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>0</td>
<td>$\frac{1}{2}$</td>
</tr>
</tbody>
</table>

In the spectrum, one observes three lines with a splitting of $\Delta \nu = 2.098 \text{ cm}^{-1}$.

c) Let $\Delta E$ be the energy splitting between the levels in the case of the Paschen-Back effect:

$$\Delta E = \mu_B B_0 = \frac{e\hbar}{2m_0} B_0 = \hbar \nu$$

then

$$\left( \frac{e}{m_0} \right) = \frac{4\pi \nu}{B_0} = 1.76 \cdot 10^{11} \text{ C/kg}$$

(Literature value: $e/m_0 = 1.7588 \cdot 10^{11} \text{ C/kg}$).

d) Since the energy-level splitting due to the Paschen-Back effect is independent of $n$, the splitting of the first line in the Lyman series is the same in wavenumbers as that of the $H_\alpha$ line. Thus $\Delta \lambda (\text{Lyman}) < \lambda (H_\alpha)$.

13.8 a), b): see Chap. 19

c): $n \geq 5$.

14.1 The Hamiltonian is

$$H = \frac{1}{2m_0} \left( p + eA \right)^2$$

$$= \frac{1}{2m_0} \left( -\hbar^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + 2eBx \frac{\hbar}{i} \frac{\partial}{\partial y} + e^2B^2x^2 \right).$$

Result:

$$E_n = \hbar \omega_L (n + \frac{1}{2})$$

$$\omega_L = \frac{|e| |B|}{2m_0}.$$ 

14.2 By substituting in the corresponding matrices, the given relations can be immediately verified.

14.3

$$\sqrt{c^2p^2 + m_0^2c^4} = m_0c^2 \sqrt{1 + \frac{p^2}{m_0^2c^2}}$$

$$\approx m_0c^2 \left( 1 + \frac{1}{2} \frac{p^2}{m_0^2c^2} \right)$$

(Power series expansion of the square root terminated after the second term.)

14.4

$$\left( \Box^2 - (m_0c/\hbar)^2 \right) \psi = 0 \text{ (Klein-Gordon Equation).}$$

From this:

$$\psi^* \left( \Box^2 - (m_0c/\hbar)^2 \right) \psi = 0.$$ 

The complex conjugate expression is

$$\psi \left( \Box^2 - (m_0c/\hbar)^2 \right) \psi^* = 0.$$

Taking the difference:

$$\psi^* \Box^2 \psi - \psi \Box^2 \psi^* = 0.$$ 

With

$$\Box^2 = \left( \frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right)$$

we have

$$\psi^* \Box^2 \psi - \psi \Box^2 \psi^*$$

$$- \frac{1}{c^2} \left( \psi^* \frac{\partial^2 \psi}{\partial t^2} - \psi \frac{\partial^2 \psi^*}{\partial t^2} \right) = 0.$$ 

From this we obtain

$$\frac{1}{c^2} \frac{\partial}{\partial t} \left( \psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t} \right)$$

$$+ \text{div} (\psi^* \nabla \psi - \psi \nabla \psi^*) = 0.$$ 

Multiplying the preceding expression by $\hbar/2im_0$, we obtain

$$\frac{\partial}{\partial t} \left[ \frac{i\hbar}{2m_0c^2} \left( \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) \right]$$

$$+ \text{div} \left[ \frac{\hbar}{2im_0} (\psi^* \nabla \psi - \psi \nabla \psi^*) \right] = 0,$$
or

\[ \frac{\partial}{\partial t} \psi + \text{div} j = 0. \]

14.5 \( (i \hbar)^2 \frac{\partial^2}{\partial t^2} \psi = i \hbar \frac{\partial}{\partial t} \mathcal{H} \psi = \mathcal{H} i \hbar \frac{\partial}{\partial t} \psi = \mathcal{H}^2 \psi, \)

\[ = (-\hbar^2 c^2 \nabla^2 + m_0^2 c^4) \psi \quad \text{or} \]

\[ \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \psi = \frac{m_0^2 c^2}{\hbar^2} \psi. \]

14.6 Substitution into the Dirac equation yields

\[
\begin{pmatrix}
-E + m_0 c^2 & 0 & ck & 0 \\
0 & -E + m_0 c^2 & 0 & -ck \\
ck & 0 & -E - m_0 c^2 & 0 \\
0 & -ck & 0 & -E - m_0 c^2
\end{pmatrix}
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\psi_3 \\
\psi_4
\end{pmatrix} = 0.
\]

Using the trial solutions \((\psi_1, 0, \psi_3, 0)\) and \((0, \psi_2, 0, \psi_4)\), one can decompose this system of equations. The corresponding energies are

\[ E = \pm \sqrt{m_0^2 c^4 + c^2 \hbar^2 k^2}. \]

15.1 Electric field strength = \( F \)

\( n = 1 \) level:

\[ \psi = \phi_1 - \bar{F} \phi_0 + \sqrt{2} \tilde{F} \phi_2 \]

\[ E^{(1)} = 0, \quad E^{(2)} = -\frac{e^2 F^2}{2m_0 \omega^2}. \]

\( n = 2 \) level:

\[ \psi = \phi_2 - \sqrt{2} \bar{F} \phi_0 + \sqrt{3} \tilde{F} \phi_3 \]

\[ E^{(1)} = 0, \quad E^{(2)} = \frac{e^2 F^2}{2m_0 \omega^2}, \quad F = -eF/\sqrt{2m_0 \omega^2}. \]

15.2 Eigenfunctions and eigenvalues:

\[ \theta_n(\phi) = \frac{1}{\sqrt{2\pi}} e^{in\phi} \quad n = 0, \pm 1, \pm 2, \pm 3 \ldots \]

\[ E_n^0 = \frac{\hbar^2 n^2}{M_0 r_0^2}. \]

First-order perturbation theory:

\[ E_n^1 = 0 \quad \text{for} \quad n \neq \pm 1 \]

for \( n = \pm 1 \):

\[ E_+ = \frac{1}{2} a, \quad \theta_+ = \frac{1}{\sqrt{4\pi}} (e^{i\phi} + e^{-i\phi}); \]

\[ E_- = -\frac{1}{2} a, \quad \theta_- = \frac{i}{\sqrt{4\pi}} (e^{i\phi} - e^{-i\phi}). \]

15.3 \( \mathcal{H}_{\text{field}} \Phi = \sum_{\lambda} \hbar \omega_{\lambda} (b_{\lambda}^+ b_{\lambda} + \frac{1}{2}) \Phi = E \Phi \)

with \( \Phi_{\lambda} = b_{\lambda}^+ \Phi_0 \), we obtain

\[ \sum_{\lambda} \hbar \omega_{\lambda} (b_{\lambda}^+ b_{\lambda} + \frac{1}{2}) \Phi_0 = E b_{\lambda}^+ \Phi_0. \]

Making use of \( b_2 b_1^+ - b_1^+ b_2 = \delta_{\lambda \lambda'} \), it follows that

\[ \hbar \omega_{\lambda} b_{\lambda}^+ \Phi_0 + \sum_{\lambda} \hbar \omega_{\lambda} b_{\lambda}^+ b_{\lambda}^+ b_{\lambda} \Phi_0 \]

\[ + \sum_{\lambda} \hbar \omega_{\lambda} (\frac{1}{2}) b_{\lambda}^+ \Phi_0 = E b_{\lambda}^+ \Phi_0. \]

Since \( b_{\lambda} \Phi_0 = 0 \), we have

\[ \hbar \omega_{\lambda} b_{\lambda}^+ \Phi_0 + \sum_{\lambda} \hbar \omega_{\lambda} (\frac{1}{2}) b_{\lambda}^+ \Phi_0 = E b_{\lambda}^+ \Phi_0. \]

Then \( \Phi_{\lambda} \) obeys expression (1) under the condition that \( E = \hbar \omega_{\lambda} + \frac{1}{2} \sum_{\lambda} \hbar \omega_{\lambda} \).

15.4 \( N \sum_{\lambda} \hbar \omega_{\lambda} b_{\lambda}^+ b_{\lambda}^+ (b_{\lambda}^+ b_{\lambda}^+) n_1 \ldots (b_{N}^+ b_{N}^+) n_N \Phi_0 \]

\[ + (N/2) \sum_{\lambda} \hbar \omega_{\lambda} (b_{\lambda}^+)^{n_1} (b_{\lambda}^+) n_2 \ldots (b_{N}^+)^{n_N} \Phi_0 \]

\[ = E N (b_{\lambda}^+)^{n_1} (b_{\lambda}^+) n_2 \ldots (b_{N}^+)^{n_N} \Phi_0, \]

with \( N = 1/\sqrt{n_1! n_2! \ldots n_N!} \).

Taking into account the commutation relations given, we obtain:

\[ N \hbar \omega_1 b_1^+ n_1 (b_1^+)^{n_1} (b_1^+) n_2 \ldots (b_{N}^+)^{n_N} \Phi_0 \]

\[ + N \hbar \omega_2 b_2^+ n_2 (b_1^+)^{n_1} (b_2^+)^{n_2} \ldots (b_{N}^+)^{n_N} \Phi_0 \]

\[ + \ldots N \hbar \omega_N b_N^+ n_N (b_1^+)^{n_1} (b_2^+)^{n_2} \ldots (b_{N}^+)^{n_N} \Phi_0 \]

\[ + \ldots N \hbar \omega_1 b_1^+ n_1 (b_1^+) n_2 \ldots (b_{N}^+)^{n_N} \Phi_0 + (N/2) \sum_{\lambda} \hbar \omega_{\lambda} (b_{\lambda}^+)^{n_1} \]

\[ \times (b_{\lambda'}^+) n_2 \ldots (b_{N}^+)^{n_N} \Phi_0 \]

\[ = E N (b_{\lambda}^+)^{n_1} (b_{\lambda}^+) n_2 \ldots (b_{N}^+)^{n_N} \Phi_0. \]
With \( b_k \Phi_0 = 0 \) and \( b_k^* (b_k^+)^n - (b_k^+)^n b_k^* = 0 \), it follows that

\[
N \sum \lambda \hbar \omega_\lambda n_\lambda (b_1^+)^{\lambda_1} (b_2^+)^{\lambda_2} \cdots (b_N^+)^{\lambda_N} \Phi_0 \\
+ \frac{N}{2} \sum \lambda \hbar \omega_\lambda (b_1^+)^{\lambda_1} (b_2^+)^{\lambda_2} \cdots (b_N^+)^{\lambda_N} \Phi_0 \\
= \frac{N}{2} \hbar \omega_\lambda (b_1^+)^{\lambda_1} (b_2^+)^{\lambda_2} \cdots (b_N^+)^{\lambda_N} \Phi_0.
\]

Then \( \Phi \) obeys relation (1) with the condition

\[
E = \sum \lambda \hbar \omega_\lambda n_\lambda + \left( \frac{1}{2} \right) \sum \hbar \omega_\lambda.
\]

15.5 \( \int \psi^*_n(r) \hat{\rho}^2 \psi_n(r) dr \)

\[
= \sum_n \int \psi^*_n(r) \hat{\rho} \cdot \hat{\rho} \psi_n(r) dr' \int \psi^*_n(r) \hat{\rho} \cdot \hat{\rho} \psi_n(r') dr' \]

\[
= \int \{ \psi^*_n(r) \hat{\rho} \cdot \{ \hat{\rho} \cdot \psi_n(r') \} \} \sum_n' \psi_n'(r) \psi^*_n(r') dr' dr \\
= \sum_n \{ \psi^*_n(r) \hat{\rho} \cdot \{ \hat{\rho} \cdot \psi_n(r') \} \} \}
\]

15.6 Rearrangements: \( \sum \langle n' | \hat{\rho} | n \rangle^2 (E_n^0 - E_n^0) \)

\[
= \sum_n \langle n | \hat{\rho} | n' \rangle \langle n' | \hat{\rho} | n \rangle (E_n^0 - E_n^0) \\
= \frac{1}{2} \sum_n \{ \langle n | \hat{\rho} | n' \rangle \langle n' | \hat{\rho} | n \rangle \} \\
+ \langle n | \hat{\rho} | n' \rangle \langle n | \hat{\rho} | n' \rangle - \langle \hat{\rho} n | \hat{\rho} | n' \rangle \}
\]

\[
J = L - S \ldots L + S.
\]

That is, the number of states is conserved in \( L - S \) coupling. (The case \( L < S \) can be obtained by exchanging \( L \) and \( S \).)

17.2 \( \sum (2J + 1) = \sum \frac{S}{4} \frac{2L + k}{2L + 1} = (S + 1)(2L + 1) \)

\[
= N
\]

Possible states:

<table>
<thead>
<tr>
<th>( S )</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L )</td>
<td>( 1 )</td>
<td>( 1 )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>( 1P_1 )</td>
<td>( 1D_2 )</td>
<td>( 1F_3 )</td>
<td></td>
</tr>
<tr>
<td>( 3P_0 )</td>
<td>( 3P_1 )</td>
<td>( 3D_2 )</td>
<td></td>
</tr>
<tr>
<td>( 3D_3 )</td>
<td>( 3F_2 )</td>
<td>( 3F_3 )</td>
<td>( 3F_4 )</td>
</tr>
</tbody>
</table>
Each term includes \((2J + 1)\) states; thus, there are altogether 60 states.

b) \(j - j\) coupling:

- \(l_1 = 1, s_1 = 1/2\) \(j_1 = 1/2, 3/2\)
- \(l_2 = 2, s_2 = 1/2\) \(j_2 = 3/2, 5/2\)

Possible states:

- \(\left(\frac{1}{2}\right) \rightarrow J = 1; 2\)
- \(\left(\frac{3}{2}\right) \rightarrow J = 0; 1; 2; 3\)
- \(\left(\frac{1}{2}\right) \rightarrow J = 2; 3\)
- \(\left(\frac{3}{2}\right) \rightarrow J = 1; 2; 3; 4\)

The same abundances of the different \(J\) values are found, and thus the same total number of states.

### 17.4

a) Electron configuration of the excited C atom:

\[
\text{C: } 1s^2 \ 2s^2 \ 2p^2 \ 3d^1 \ l_2 = 2, \ \ \ l_1 = 1, \ \ \ s_2 = 1/2 \ \ \ s_1 = 1/2
\]

Possible \(L\) and \(S\) values:

- \(L = 1, 2, 3; S = 0, 1\)

Possible \(J\) values:

- \(J = 0, 1, 2, 3, 4\)

Possible terms:

- \(^1P_{1,2}^1D_1^1F_3\) singlet \(S = 0\)
- \(^3P_{0,2}^3D_1^3F_2\) triplet \(S = 1\)

b) Occupation of the \(3d\) subshell due to Hund’s rules (cf. Sect. 19.2):

\[
\begin{array}{cccc}
3d & \uparrow & \uparrow & \uparrow \\
\end{array}
\]

- \(m_l = -2, -1, 0, 1, 2\)
- \(L = 3; S = 3/2; J = 3/2 \rightarrow ^4F_{3/2}\)

Magnetic moment [cf. (13.17)]:

\[(\mu_J)_{f} = \sqrt{\frac{3}{5}} \mu_B\]

c) Occupation according to Hund’s rule (cf. Sect. 19.2) gives for the ground state in

- \(Y: L = 2, S = 1/2, J = 3/2 \rightarrow ^2D_{3/2}\)
- \(Zr: L = 3, S = 1, J = 2 \rightarrow ^3F_2\).

d) Electron configuration of Mn:

\[
\text{Mn: } 1s^2 \ 2s^2 \ 2p^6 \ 3s^2 \ 3p^6 \ 4s^2 \ 3d^5
\]

\[
\begin{array}{cccc}
3d & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
\end{array}
\]

- \(m_l = -2, -1, 0, 1, 2\)

Ground state: \(L = 0, S = 5/2, J = 5/2 \rightarrow ^6S_{5/2}\).

### 17.5

a) General procedure:

1) Determination of \(L\) and \(S\) from the term symbol;
2) Determination of \(J\) from the number \(m\) of particle beams in the Stern-Gerlach experiment: \(m = 2J + 1\);
3) Maximum magnetic moment in the direction of the magnetic field:

\[
\mu_{z,\text{max}} = g_J \mu_B;
\]

\[
g_J = 1 + \frac{J(J + 1) + S(S + 1) - L(L + 1)}{2J(J + 1)}
\]

<table>
<thead>
<tr>
<th>Atom</th>
<th>Term symbol</th>
<th>(L)</th>
<th>(S)</th>
<th>(J)</th>
<th>(g_J)</th>
<th>(\mu_{z,\text{max}}/\mu_B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vanadium</td>
<td>(^4F)</td>
<td>3</td>
<td>1/2</td>
<td>1/2</td>
<td>5/3</td>
<td>3/5</td>
</tr>
<tr>
<td>Manganese</td>
<td>(^6S)</td>
<td>0</td>
<td>3/2</td>
<td>1/2</td>
<td>5/3</td>
<td>2/3</td>
</tr>
<tr>
<td>Iron</td>
<td>(^5P)</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>3/2</td>
<td>6</td>
</tr>
</tbody>
</table>

b) For \(S = 0\), the energy level splitting in a magnetic field is given by:

\[
\Delta E = E(m_L = L) - E(m_L = -L) = 2L \mu_B B;
\]

thus,

\[
L = \frac{\Delta E}{2 \mu_B B} = \frac{h c \bar{v}}{2 \mu_B B} = 3, \quad \text{giving } ^1F_3.
\]

### 18.1

\(\lambda_{\text{min}} = 3.09 \cdot 10^{-11} \text{ m} = 0.309 \text{ Å}\)

### 18.2

\(\Delta E_{\text{Co}} = h c / \lambda = 1.114 \cdot 10^{-15} \text{ J} = 6952 \text{ eV}\)

In comparison to the hydrogen atom, we have

\[
\Delta E_{\text{H}} = h c R_{\text{H}} \left( \frac{1}{1^2} - \frac{1}{2^2} \right) = 10.21 \text{ eV} \approx \frac{\Delta E_{\text{Co}}}{(Z_{\text{Co}} - 1)^2}
\]

(cf. Moseley’s rule, Sect. 18.4).

### 18.3

\(\lambda_{K_{\alpha}}(\text{Cu}) = 1.55 \cdot 10^{-10} \text{ m} = 1.55 \text{ Å}\)

### 18.4

Manganese (\(Z = 25\))

### 18.5

Linear absorption coefficient \(\mu\):

\[
I(x) = I_0 e^{-\mu x}; \quad \mu = \frac{\ln(I_0/I(x))}{x}.
\]
492 Solutions to the Problems

$I_0$ and $I(x)$ are the intensities of the incident beam and of the beam after the distance $x$. From the experimental data it follows that

$$\bar{\mu} = (131.7 \pm 0.3) \, m^{-1}.$$  

### 18.6

$I = I_0 e^{-\mu x} \rightarrow x = \frac{\ln I_0/I}{\mu}$

<table>
<thead>
<tr>
<th>$h\nu$ [MeV]</th>
<th>0.05</th>
<th>0.3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$ [1/m]</td>
<td>8000</td>
<td>500</td>
<td>78</td>
</tr>
<tr>
<td>$x$ [m]</td>
<td>2.88 \cdot 10^{-4}</td>
<td>4.61 \cdot 10^{-3}</td>
<td>2.95 \cdot 10^{-2}</td>
</tr>
</tbody>
</table>

\[a) h\nu [MeV] \quad 0.05 \quad 0.3 \quad 1 \]

\[b) h\nu [MeV] \quad 0.05 \quad 0.3 \quad 1 \]

$I/I_0$ \quad 4.25 \cdot 10^{-18} \quad 8.21 \cdot 10^{-2} \quad 0.677$

### 18.7

The half-absorption length of a material:

$I = I_0 e^{-\mu d_{1/2}} = \frac{1}{2} I_0$; thus, $d_{1/2} = \frac{\ln 2}{\mu}$.

When $I_0/I(x)$ is known, it follows for the thickness $x$ that

$$x/d_{1/2} = \frac{\ln (I_0/I)}{\ln 2}.$$  

<table>
<thead>
<tr>
<th>$I_0/I$</th>
<th>16</th>
<th>20</th>
<th>200</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x/d_{1/2}$</td>
<td>4.00</td>
<td>4.32</td>
<td>7.64</td>
</tr>
</tbody>
</table>

### 18.8

a) Photon energies of the K series:

$$\epsilon_n = h\nu_n = hcR (Z-1)^2 \left( \frac{1}{1^2} - \frac{1}{n^2} \right).$$

Energy levels:

<table>
<thead>
<tr>
<th>Shell</th>
<th>$K$</th>
<th>$L$</th>
<th>$M$</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy [eV]</td>
<td>-69660</td>
<td>-10620</td>
<td>-2280</td>
<td>-390</td>
</tr>
</tbody>
</table>

b) In order to excite the L series, an electron must be removed from the L shell, i.e. one requires at least the energy

$$-E_L = -10.62 \, \text{keV}.$$  

The wavelength of the $L_\alpha$ line is $\lambda_{L_\alpha} = 1.49 \, \text{Å}.$

### 18.9

$E_{\text{kin}} = E_1 - 2E_2 = 5.57 \cdot 10^{-15} \, J = 34.8 \, \text{keV}.$

The velocity of the electron is calculated from the relativistic relation to be

$$v = c \sqrt{1 - \left( \frac{m_0c^2}{E_{\text{tot}}} \right)^2} = 1.05 \cdot 10^8 \, \text{m/s}.$$  

### 18.1

a) 15 electrons: phosphorous

b) 46 electrons: palladium

### 19.2

Consider an atom with several valence electrons having the same $n$, $l$ quantum numbers, and neglect the spin-orbit interaction and interactions of the electrons among themselves. Then the total state of the valence electrons can be represented as a product of one-electron states. A one-electron state is characterized by the quantum numbers $n$, $l$, $m_l$, and $m_s$.

$$\Psi_{\text{tot}} = \Psi_{m_1, m_{s_1}} (1) \times \Psi_{m_2, m_{s_2}} (2) \ldots$$  

The operators $\hat{L}_Z$ and $\hat{S}_Z$ for the Z components of the total orbital angular momentum and the total spin are

$$\hat{L}_Z = \hat{i}_{Z_1} + \hat{i}_{Z_2} + \ldots; \quad \hat{S}_Z = \hat{s}_{Z_1} + \hat{s}_{Z_2} + \ldots$$

If a shell $n$, $l$ is fully occupied, all of the possible one-electron states $(m_l, m_s)$ are filled just once, according to the Pauli principle. From (1), the following wavefunction is obtained:

$$\Psi_{\text{tot}} = \Psi_{L, +1/2} (1) \times \Psi_{L, -1/2} (2) \ldots$$

$$\cdot \Psi_{-L, +1/2} (4L+1) \times \Psi_{-L, -1/2} (4L+2).$$  

$\Psi_{\text{tot}}$ is an eigenfunction of $\hat{L}_Z$ and of $\hat{S}_Z$. The eigenvalues are $m_l$, $h = 0$ and $m_s$, $h = 0$.

Therefore, the total orbital angular momentum and the total spin must vanish, i.e. $L = S = 0$.

Note: the function $\Psi_{\text{tot}}$ in (1) does not yet fulfill the Pauli principle, since it is not antisymmetric with respect to exchange of two electrons. Strictly speaking, one would have to set up a determinant as in (19.18); this has no effect on the conclusions drawn above.

### 19.3

Pauli Principle:

The state of an electron in an atom can be specified by four quantum numbers. If the spin-orbit coupling and the interaction with other electrons which may be present are neglected, the quantum
numbers \( n, \ell, m_l, \) and \( m_s \) may be used. The Pauli principle then requires that in a system containing several electrons, no two electrons may have exactly the same four quantum number values. This principle is based on the indistinguishability of electrons.

If one uses the quantum numbers \( n, \ell, m_l, \) and \( m_s, \) then, taking the Pauli principle into account, the following two-electron states are found (\( \times \)) for an \( np^2 \) configuration:

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>((m_1, m_s)_1)</td>
</tr>
<tr>
<td>((m_1, m_s)_2)</td>
</tr>
<tr>
<td>((1, +))</td>
</tr>
<tr>
<td>((1, -))</td>
</tr>
<tr>
<td>((0, +))</td>
</tr>
<tr>
<td>((0, -))</td>
</tr>
<tr>
<td>((-1, +))</td>
</tr>
<tr>
<td>((-1, -))</td>
</tr>
</tbody>
</table>

In the following, we denote such a two-particle state by the expression \([(m_1, m_s_1)(m_2, m_s_2)]\) with \( m_1 = \pm 1, 0, -1 \) and \( m_s = \pm (1/2) \).

The coupling of two angular momenta:

When two arbitrary angular momenta \( J_1 \) and \( J_2 \) are coupled to a total angular momentum \( J = J_1 + J_2 \), the quantum number \( J \) can take on the values \( J = J_1 + J_2, \ldots, |J_1 - J_2| \). For each \( J \) value—there are the \( m_J \) values \( m_J = -J, \ldots, +J \).

The corresponding total angular momentum state \( (J, m_J) \) can be represented as a linear combination of the product states \( (J_1, m_{J_1}) \cdot (J_2, m_{J_2}) \), where \( m_J = m_{J_1} + m_{J_2} \) is always fulfilled.

\[
(J, m_J) = \sum_{m_{J_1}, m_{J_2}} C_{m_{J_1}, m_{J_2}} (J_1, m_{J_1}) \cdot (J_2, m_{J_2})
\]

(1)

After this coupling, \( m_{J_1} \) and \( m_{J_2} \) are no longer valid quantum numbers. An angular momentum state is specified uniquely by the quantum numbers \( J_1, J_2, J, \) and \( m_J \). In both \( j-j \) and in \( L-S \) coupling, one arrives at the total angular momentum \( J \) of the system by means of such a two-stage angular momentum coupling, starting from the non-interacting particle model.

a) \( j-j \) Coupling:

In the case of \( j-j \) coupling, in the first step, the orbital angular momentum \( l \) and the spin \( s \) of each particle (electron) are coupled to a single-electron angular momentum \( j \). Table 2 shows the ordering of all the states which are consistent with the Pauli principle from Table 1, according to the new single-particle quantum number \( m_j \).

<table>
<thead>
<tr>
<th>Table 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_{J_1} )</td>
</tr>
<tr>
<td>---------</td>
</tr>
<tr>
<td>( 3/2 )</td>
</tr>
<tr>
<td>( 1/2 )</td>
</tr>
<tr>
<td>(-1/2 )</td>
</tr>
<tr>
<td>(-3/2 )</td>
</tr>
</tbody>
</table>

In the second step, the single-particle angular momenta are coupled to give the total angular momentum \( J(j-j \) coupling). From (1), only those states contribute to a state with quantum number \( m_J \) (belonging to \( J \)) for which \( m_{J_1} + m_{J_2} = m_J \). The number of such states is conserved, so the abundance of the various possible \( m_J \) values may be calculated.

| \( m_J \) | \( 2 \) | \( 1 \) | \( 0 \) | \(-1 \) | \(-2 \) |
|--------|-------|-------|-------|-------|
| Abundance | \( 2 \) | \( 3 \) | \( 5 \) | \( 3 \) | \( 2 \) |

Here, the following quantum numbers \( J \) occur:

| \( J \) | \( 2 \) | \( 1 \) | \( 0 \) |
|--------|-------|-------|
| Abundance | \( 2 \) | \( 1 \) | \( 2 \) |

b) \( L-S \) Coupling:

In the first step, the orbital angular momenta \( l_1 \) and \( l_2 \) couple to give the total orbital angular momentum \( L \) and the spins \( s_1 \) and \( s_2 \) couple to the total spin \( S \). Table 3 gives the ordering of the states which are consistent with the Pauli principle from Table 1, according to the new quantum numbers \( m_S \) and \( m_L \).
Table 3

<table>
<thead>
<tr>
<th>m_L</th>
<th>m_S</th>
<th>m_j</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>-2</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

leading to the possible terms:

<table>
<thead>
<tr>
<th>Terms</th>
<th>Number of states</th>
</tr>
</thead>
<tbody>
<tr>
<td>0_s</td>
<td>2</td>
</tr>
<tr>
<td>1_1/2</td>
<td>3</td>
</tr>
<tr>
<td>1_1</td>
<td>3</td>
</tr>
<tr>
<td>1_0</td>
<td>3</td>
</tr>
<tr>
<td>2_1/2</td>
<td>3</td>
</tr>
<tr>
<td>2_1</td>
<td>3</td>
</tr>
<tr>
<td>2_0</td>
<td>3</td>
</tr>
<tr>
<td>3_1/2</td>
<td>3</td>
</tr>
<tr>
<td>3_1</td>
<td>3</td>
</tr>
</tbody>
</table>

For the L - S coupling to a total angular momentum J we find, similarly to part (a):

<table>
<thead>
<tr>
<th>m_j</th>
<th>2</th>
<th>1</th>
<th>0</th>
<th>-1</th>
<th>-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abundance</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

and from this:

<table>
<thead>
<tr>
<th>J</th>
<th>2</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abundance</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Thus, the same J values occur with the same abundances.

19.4 The given electron configurations are considered in the light of L - S coupling. If several electrons have the same n and l values, the Pauli principle must be observed (cf. Problem 19.3).

a) Electron configuration ns:
Possible term: ^2S_1/2 → 2 states

b) Electron configuration np^3:
One finds the following abundances for the various (m_L, m_S) pairs:

c) Electron configuration np^2 n's:
Possible terms:

<table>
<thead>
<tr>
<th>L</th>
<th>S</th>
<th>Terms</th>
<th>Number of states</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1/2</td>
<td>^2D_3/2</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>1/2</td>
<td>^2P_1/2</td>
<td>6</td>
</tr>
<tr>
<td>0</td>
<td>3/2</td>
<td>^4S_3/2</td>
<td>4</td>
</tr>
</tbody>
</table>


d) Electron configuration np^5:
Possible terms: L = 1; S = 1/2: ^2P_1/2 ^2P_3/2; 6 states

e) Electron configuration nd^2 n'p:
Possible terms:

<table>
<thead>
<tr>
<th>L</th>
<th>S</th>
<th>Terms</th>
<th>Number of states</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1/2</td>
<td>^2H_9/2</td>
<td>22</td>
</tr>
<tr>
<td>4</td>
<td>1/2</td>
<td>2x (^2G_7/2 ^2G_9/2)</td>
<td>36</td>
</tr>
<tr>
<td>3</td>
<td>3/2</td>
<td>^4G_5/2</td>
<td>36</td>
</tr>
<tr>
<td>3</td>
<td>1/2</td>
<td>3x (^2F_5/2 ^2F_7/2)</td>
<td>42</td>
</tr>
<tr>
<td>2</td>
<td>3/2</td>
<td>^4F_3/2</td>
<td>42</td>
</tr>
<tr>
<td>1</td>
<td>1/2</td>
<td>3x (^2D_3/2 ^2D_5/2)</td>
<td>30</td>
</tr>
<tr>
<td>0</td>
<td>1/2</td>
<td>^4P_1/2</td>
<td>18</td>
</tr>
<tr>
<td>0</td>
<td>3/2</td>
<td>^4S_3/2</td>
<td>12</td>
</tr>
</tbody>
</table>
f) Electron configuration \textit{nd nd'}:
Possible terms:
\begin{tabular}{|c|c|c|c|}
\hline
\textit{L} & \textit{S} & Terms & Number of states \\
\hline
4 & 0 & \textit{1}G_{4} & 9 \\
1 & 3 & \textit{3}G_{3} \textit{3}G_{4} \textit{3}G_{5} & 27 \\
3 & 0 & \textit{1}F_{3} & 7 \\
1 & 3 & \textit{3}F_{3} \textit{3}F_{3} \textit{3}F_{4} & 21 \\
2 & 0 & \textit{1}D_{2} & 5 \\
1 & 3 & \textit{3}D_{1} \textit{3}D_{2} \textit{3}D_{3} & 15 \\
1 & 0 & \textit{1}P_{1} & 3 \\
1 & 3 & \textit{3}P_{0} \textit{3}P_{1} \textit{3}P_{2} & 9 \\
0 & 0 & \textit{1}S_{0} & 1 \\
1 & 3 & \textit{3}S_{1} & 3 \\
\hline
\end{tabular}

According to Hund's rules, for the ground state term we have \textit{L} = 3, \textit{S} = 1, \textit{J} = 2; i.e. \textit{3}F_{2}.

19.8 Proof of relation (19.25):
\[\Sigma \phi_{1}(1) \phi_{1}(2) = (\sigma_{z,1} + \sigma_{z,2}) \phi_{1}(1) \phi_{1}(2) = (\frac{1}{2} \hbar + \frac{1}{2} \hbar) \phi_{1}(1) \phi_{1}(2).\]
The proofs of relations (19.26) and (19.32) are analogous. To prove relations (19.35–38), we require the following identities:
\[\Sigma^{2} = \sigma_{1}^{2} + 2 \sigma_{1} \sigma_{2} + \sigma_{2}^{2},\]
\[\sigma_{1} \cdot \sigma_{2} = \sigma_{z,1} \sigma_{z,2} + \sigma_{x,1} \sigma_{x,2} + \sigma_{y,1} \sigma_{y,2};\]
\[\sigma_{x,1} \sigma_{x,2} + \sigma_{y,1} \sigma_{y,2} = \frac{1}{2} (\sigma_{+,1} - \sigma_{-,1} \sigma_{+,2} + \sigma_{-,1} \sigma_{+,2}).\]
With these, it follows that
\[\Sigma^{2} = \sigma_{1}^{2} + 2 \sigma_{1} \sigma_{2} + \sigma_{2}^{2},\]
and the following relations are valid:
\[\sigma_{n}^{2} \phi_{1}(1) = \frac{1}{4} \hbar \phi_{1}(1), \sigma_{n}^{2} \phi_{1}(1) = \frac{1}{4} \hbar \phi_{1}(1);\]
\[\sigma_{+,1} \phi_{1}(1) = 0, \sigma_{+,1} \phi_{1}(1) = h \phi_{1}(1);\]
\[\sigma_{-,1} \phi_{1}(1) = 0, \sigma_{-,1} \phi_{1}(1) = h \phi_{1}(1);\]
\[\sigma_{z,1} \phi_{1}(1) = h \phi_{1}(1), \sigma_{z,1} \phi_{1}(1) = -h \phi_{1}(1).\]
(similar relations hold for the second electron.)
With these, one can prove (19.35–38).

19.9 When the quantum numbers \textit{q} and \textit{q} are identical, no triplet state exists.
[See (19.19, 20 and 30)]

20.1 The formula for hyperfine structure (20.10) is
\[\Delta E_{F} = \frac{a}{2} [F(F+1) - I(I+1) - J(J+1)].\]
The interval rule (20.12) states that
\[\Delta E_{F+1} - \Delta E_{F} = a(F+1).\]
From this, \textit{F}_{\text{max}} and \textit{a} may be determined to be
\[F_{\text{max}} = 7, \text{ } a = (0.0789 \pm 0.0006) \text{ cm}^{-1},\]
and \textit{F} = 2, ..., 7 so that \textit{I} = 9/2.

\begin{tabular}{|c|c|c|c|c|}
\hline
\textit{F} & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
\textit{\Delta E_{F}/a} & -13.75 & -10.75 & -6.75 & -1.75 & 4.25 & 11.25 \\
\hline
\end{tabular}
20.2 The hyperfine structure formula (20.10) is

\[ \Delta E_F = \frac{a}{2} [F(F+1) - I(I+1) - J(J+1)]; \]

\[ a = \frac{g_I \mu_B B_I}{\sqrt{J(J+1)}}. \]

In the hydrogen atom, (20.11) is valid:

\[ a = \frac{2}{3} \mu_0 g_e \mu_B |\psi(0)|^2; |\psi(0)|^2 = \frac{1}{\pi r_H^3 n^3}. \]

(Here, \( r_H \) is the first Bohr radius.) Then

\[ B_I = \frac{2}{3} \mu_0 g_e \mu_B |\psi(0)|^2 \sqrt{J(J+1)}. \]

According to the interval rule, the energy difference between parallel (\( F=1 \)) and antiparallel (\( F=0 \)) orientations of the electronic and the proton spins is just \( a \).

| \( n \) | \( |\psi(0)|^2 \) [1/m^3] | \( B_I \) [T] | \( a \) [cm^{-1}] |
|---|---|---|---|
| 1 | 2.15 \cdot 10^{30} | 29.0 | 0.0475 |
| 2 | 2.69 \cdot 10^{29} | 3.63 | 0.00594 |
| 3 | 7.96 \cdot 10^{28} | 1.07 | 0.00176 |

20.3 \( a/\hbar c = (5.85 + 0.52) \cdot 10^{-4} \) cm^{-1}

\( B_I = (3.03 \pm 0.27) \) T

20.4 Zeeman energy of the electron:

\[ E_e^- = g_e \mu_B B_0 m_s = 5.57 \cdot 10^{-24} \text{ J} \cdot m_s \]

\[ = 3.48 \cdot 10^{-5} \text{ eV} \cdot m_s = h c \cdot 0.281 \text{ cm}^{-1} \cdot m_s. \]

Hyperfine interaction energy with \( a = 0.0475 \) cm^{-1}:

\[ E_{\text{HFS}} = a m_I m_s = h c \cdot 0.0475 \text{ cm}^{-1} \cdot m_I m_s \]

\[ = 5.89 \cdot 10^{-6} \text{ eV} \cdot m_I m_s \]

\[ = 9.44 \cdot 10^{-25} \text{ J} \cdot m_I m_s. \]

Zeeman energy of the nucleus (\( g_I = 5.585 \)):

\[ E_p = -g_I \mu_N B_0 m_I = -8.46 \cdot 10^{-27} \text{ J} \cdot m_I \]

\[ = -5.28 \cdot 10^{-8} \text{ eV} \cdot m_I = h c \cdot 4.26 \cdot 10^{-4} \text{ cm}^{-1} \cdot m_I. \]

20.5 \( \gamma = 6.624 \cdot 10^7 \) m^2/Vs^2

\( g_I = 1.38 \)

\[ \frac{\mu_z, \text{max}}{\mu_N} = 3.45 \]

20.6 According to (20.15), in the case of a “strong” applied magnetic field we have

\[ \Delta E_{\text{HFS}} = g_I \mu_B m_I B_1 + a m_I m_J. \]

In the Stern-Gerlach experiment, the beam splits into \((2J+1)(2J+1)\) sub-beams; thus, we have \( I^{(23} \text{Na}) = 3/2. \)

In the case of a “weak” magnetic field, the beam is also split into 8 sub-beams, which, however, have a different geometrical arrangement.

21.1 \( W = 1.224 \cdot 10^{-12} \text{ s}^{-1} \)

21.2

<table>
<thead>
<tr>
<th>( L \rightarrow )</th>
<th>1 cm</th>
<th>10 cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R = 99% )</td>
<td>( t_0 = 3.336 \cdot 10^{-9} ) s</td>
<td>( t_0 = 3.336 \cdot 10^{-8} ) s</td>
</tr>
<tr>
<td>( R = 90% )</td>
<td>( t_0 = 3.336 \cdot 10^{-10} ) s</td>
<td>( t_0 = 3.336 \cdot 10^{-9} ) s</td>
</tr>
<tr>
<td>( R = 10% )</td>
<td>( t_0 = 3.706 \cdot 10^{-11} ) s</td>
<td>( t_0 = 3.706 \cdot 10^{-10} ) s</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( L \rightarrow )</th>
<th>100 cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R = 99% )</td>
<td>( t_0 = 3.336 \cdot 10^{-7} ) s</td>
</tr>
<tr>
<td>( R = 90% )</td>
<td>( t_0 = 3.336 \cdot 10^{-8} ) s</td>
</tr>
<tr>
<td>( R = 10% )</td>
<td>( t_0 = 3.706 \cdot 10^{-9} ) s</td>
</tr>
</tbody>
</table>

21.3 \( R = 90\%, L = 100 \text{ cm}: \)

The number of photons decreases to \( 1/e \) of its initial value after a time \( T = 3.336 \cdot 10^{-8} \) s.

21.4 The laser condition is

\[ \frac{N_2 - N_1}{V} < \frac{8 \pi v^2 \Delta v \tau}{c^3 t_0}. \]

The formula

\[ \frac{1}{t_0} = \frac{c}{L} (1 - R) \]

gives the lifetime of the photons in the resonator.

For the length of the resonator, we assume e.g. 10 cm, and thus obtain the value \( 1.2 \cdot 10^{18}/\text{cm}^3 \) for the critical inversion density.

22.1 Inserting (***) into (*) and taking into account the rules given yields equations of the form (14.87) and (14.88), which can then be treated further as in Chap. 14.
23.1 One requires only the orthogonality of the wavefunctions.

23.2 $\psi(1,2)$

$$= \frac{1}{\sqrt{2}} \{(d_1| \downarrow \rangle_1 + d_2| \uparrow \rangle_1)(d_1^*| \downarrow \rangle_2 + d_2^*| \uparrow \rangle_2)$$

$$+ (d_2^*| \downarrow \rangle_1 - d_1^*| \uparrow \rangle_1)(d_2| \downarrow \rangle_2 - d_1| \uparrow \rangle_2)\}$$

$$= \frac{1}{\sqrt{2}} \{|d_1|^2 \downarrow \rangle + |d_2|^2 \uparrow \rangle + |d_2|^2 \downarrow \rangle$$

$$+ (|d_1|^2 \uparrow \rangle)\}.$$

23.3 We make use of the spin formalism and compute:

$$\frac{1}{2} (\langle \uparrow | A \downarrow \rangle B - \langle \downarrow | A \uparrow \rangle B)$$

$$= \frac{1}{2} (\langle \uparrow | \hat{A} \downarrow \rangle | \uparrow \rangle)$$

$$- (\langle \uparrow | \hat{A} \downarrow \rangle | \downarrow \rangle)$$

$$= a_{ix} \langle \uparrow | \sigma_x \uparrow \rangle + a_{iy} \langle \uparrow | \sigma_y \uparrow \rangle + a_{iz} \langle \uparrow | \sigma_z \uparrow \rangle$$

$$= a_{iz}$$

due to the properties of the spin matrices.

23.4 1) $e_1 = | \uparrow \rangle$, $e_2 = | \downarrow \rangle$ are orthonormalised.

2) All vectors $v$ in the Hilbert space can be expressed as $v = ae_1 + be_2$, where $|a|^2 + |b|^2 = 1$.

Due to 1), these vectors are also normalised.

23.5 The photon plays the role of controller.

24.1 Transform the coordinates according to $x_b = -x_a$, $x_a > 0$.

$$\psi(x) = A e^{\kappa x} \quad x \leq -x_a$$

$$\psi(x) = B e^{\kappa x} + C e^{-\kappa x} \quad -x_a \leq x \leq +x_a$$

$$\psi(x) = D e^{-\kappa x} \quad x \geq x_a$$

$$\kappa = \sqrt{2m_0|E|/h^2}$$

Symmetric state: $\psi(x) = \psi(-x)$.

$$\kappa - m_0 \beta/h^2 = e^{-2\kappa x_a m_0 \beta/h^2}$$

$B = C$, $A = D$

Antisymmetric state: $\psi(x) = -\psi(-x)$.

$$\kappa - m_0 \beta/h^2 = -e^{-2\kappa x_a m_0 \beta/h^2}$$

$B = -C$, $A = -D$

24.2 Transform the coordinates, so that $x_b = -x_a$, $x_a > 0$.

$$\psi(x) = A e^{\kappa x} \quad x \leq -x_a$$

$$\psi(x) = B e^{\kappa x} + C e^{-\kappa x} \quad -x_a \leq x \leq +x_a$$

$$\psi(x) = D e^{-\kappa x} \quad x \geq x_a$$

$$\kappa = \sqrt{2m_0|E|/h^2}$$

$$(x - m_0 \beta_a/h^2) = e^{-2\kappa x_a m_0 \beta/h^2}$$

$$B = C/B$$

Antisymmetric state: $\psi(x) = -\psi(-x)$.

$$\kappa - m_0 \beta/h^2 = -e^{-2\kappa x_a m_0 \beta/h^2}$$

$B = -C$, $A = -D$

24.3 Wavefunction:

$$\psi_k(x) = e^{inka} (A_k e^{\kappa(x - na)} + B_k e^{-\kappa(x - na)})$$

$$\text{for } na \leq x \leq (n + 1)a$$

$$A_k = B_k (e^{ikx} - e^{-x}) (e^{xa} - e^{-xa})$$

$$\kappa = \sqrt{2m_0|E|/h^2}$$

$$ka = \frac{2\pi}{6}, \frac{2\pi}{6}, \ldots, \frac{2\pi}{6}, \ldots$$

$$\text{Auxiliary equation for } \kappa:$$

$$\cos ka = \cosh \kappa a - \frac{\sinh \kappa a}{\kappa} \cdot m_0 \beta/h^2.$$
Bibliography of Supplementary and Specialised Literature

1. General Physics Textbooks

M. Alonso, E.J. Finn: Physics (Addison-Wesley, Reading 1992)
P.A. Tipler: Physics for Scientists & Engineers (Worth, New York 1999)

2. Atomic Physics Textbooks

M. Born: Atomic Physics, 8th ed. (Dover, New York 1989)

3. Quantum Mechanics Textbooks


4. Specialised Literature

Chapter 1

E. Segrè: *From x-Rays to Quarks. Modern Physicists and their Discoveries* (Freeman, New York 1980)
B. L. van der Waerden (ed.): *Sources of Quantum Mechanics* (Dover, New York 1967)

Chapter 2

C. Kittel, H. Kroemer: *Thermal Physics*, 2nd ed. (Freeman, San Francisco 1980)

Chapter 3

S. Villani (ed.): *Uranium Enrichment* (Springer, Berlin Heidelberg 1979)
Chapter 4
E. Segre: *Nuclei and Particles: An Introduction to Nuclear & Subnuclear Physics*, 2nd ed. (Benjamin, Reading 1977)

Chapter 5

Chapter 6 Atomic Physics Textbooks

Chapter 7 Quantum Mechanics Textbooks

Chapter 8
B. L. van der Waerden (ed.): *Sources of Quantum Mechanics* (Dover, New York 1967)

Chapters 9, 10 Quantum Mechanics Textbooks

Chapters 11, 12

Chapter 13, 14

**Section 14.7**


**Chapter 15**


**Chapter 16**


**Chapter 17**


G. Herzberg: *Atomic Spectra and Atomic Structure* (Dover, New York 1944)


**Chapter 18**


**Chapter 19**


**Chapter 20**


**Chapter 21**


**Chapter 22**

A. Corney: *Atomic and Laser Spectroscopy* (Oxford University Press 1987)

**Section 22.2**


**Section 22.7**

V. B. Braginsky, F. Y. Khalili: *Quantum Measurement* ed. by K. S. Thorne (Cambridge Univ. Press 1992)


Section 23.5

E. Schrödinger: Die Naturwissenschaften 48, 807 (1935) (concept of entanglement)

Section 23.6


Section 23.7

J. S. Bell: Physics (Long Island City, N.Y.) 1, 195 (1964)

Overview:

Section 23.8


Section 23.9


Error-tolerant algorithms:

Overview:

A good introduction to quantum computers:
D. Bouwmeester et al. (eds.): The Physics of Quantum Information: Quantum Cryptography, Quantum Teleportation, Quantum Computation (Springer, Berlin Heidelberg 2000)
Section 23.10


Section 23.11

Bose-Einstein condensation, exp. observation:

Interference:

Overview articles on theory:

Chapter 24

### Subject Index

<table>
<thead>
<tr>
<th>Term</th>
<th>Page(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abbe's theory</td>
<td>20</td>
</tr>
<tr>
<td>Absorption</td>
<td>13, 55, 96, 102</td>
</tr>
<tr>
<td>- in the laser mechanism</td>
<td>390, 394</td>
</tr>
<tr>
<td>Absorption coefficient</td>
<td>38, 330</td>
</tr>
<tr>
<td>Absorption edge</td>
<td>330</td>
</tr>
<tr>
<td>Actinides</td>
<td>27, 344</td>
</tr>
<tr>
<td>Adiabatic invariance principle</td>
<td>339</td>
</tr>
<tr>
<td>Alkali atoms</td>
<td></td>
</tr>
<tr>
<td>- general</td>
<td>27, 344</td>
</tr>
<tr>
<td>- level scheme</td>
<td>196f.</td>
</tr>
<tr>
<td>- spectra</td>
<td>171ff.</td>
</tr>
<tr>
<td>- term diagrams</td>
<td>172</td>
</tr>
<tr>
<td>$\alpha$ particles</td>
<td>10, 39</td>
</tr>
<tr>
<td>Amplitude fluctuations in the laser</td>
<td>396</td>
</tr>
<tr>
<td>Amplitude stability in lasers</td>
<td>396</td>
</tr>
<tr>
<td>Ancillary qubit</td>
<td>431</td>
</tr>
<tr>
<td>Angular momentum</td>
<td>115, 205</td>
</tr>
<tr>
<td>- addition of</td>
<td>314ff.</td>
</tr>
<tr>
<td>- commutation relations</td>
<td>141, 157</td>
</tr>
<tr>
<td>- conservation of</td>
<td>42</td>
</tr>
<tr>
<td>- coupling mechanism</td>
<td>314</td>
</tr>
<tr>
<td>- creation and annihilation operators</td>
<td>157</td>
</tr>
<tr>
<td>- decoupling</td>
<td>371</td>
</tr>
<tr>
<td>- eigenfunctions</td>
<td>155ff.</td>
</tr>
<tr>
<td>- in spherical polar coordinates</td>
<td>160f.</td>
</tr>
<tr>
<td>- of light quanta</td>
<td>213</td>
</tr>
<tr>
<td>- quantum number</td>
<td>164f., 186</td>
</tr>
<tr>
<td>- vector model</td>
<td>314</td>
</tr>
<tr>
<td>Annihilation</td>
<td>121</td>
</tr>
<tr>
<td>Annihilation operator</td>
<td></td>
</tr>
<tr>
<td>- for spherical harmonic functions</td>
<td>157</td>
</tr>
<tr>
<td>- for the harmonic oscillator</td>
<td>144, 279</td>
</tr>
<tr>
<td>Antihydrogen</td>
<td>120</td>
</tr>
<tr>
<td>Antineutrino</td>
<td>109</td>
</tr>
<tr>
<td>Antiparticles</td>
<td>120</td>
</tr>
<tr>
<td>Antiproton</td>
<td>120</td>
</tr>
<tr>
<td>Antisymmetry of wavefunctions</td>
<td>346, 351f.</td>
</tr>
<tr>
<td>Areas, law of</td>
<td>115</td>
</tr>
<tr>
<td>Aston's mass spectrograph</td>
<td>72</td>
</tr>
<tr>
<td>Atom</td>
<td></td>
</tr>
<tr>
<td>- angular momentum, total</td>
<td>364</td>
</tr>
<tr>
<td>- as a wave</td>
<td>76</td>
</tr>
<tr>
<td>- general</td>
<td>1, 5</td>
</tr>
<tr>
<td>- ground state</td>
<td>344</td>
</tr>
<tr>
<td>- heavy</td>
<td>319</td>
</tr>
<tr>
<td>- in an electric field</td>
<td>261ff.</td>
</tr>
<tr>
<td>- in a magnetic field</td>
<td>205ff.</td>
</tr>
<tr>
<td>- laser</td>
<td>436ff.</td>
</tr>
<tr>
<td>- mass of</td>
<td>28</td>
</tr>
<tr>
<td>- absolute</td>
<td>6</td>
</tr>
<tr>
<td>- - relative</td>
<td>5f., 29, 33</td>
</tr>
<tr>
<td>- - paramagnetic</td>
<td>206</td>
</tr>
<tr>
<td>- - radius</td>
<td>15, 20</td>
</tr>
<tr>
<td>- - shell structure</td>
<td>323</td>
</tr>
<tr>
<td>- - size</td>
<td>10, 15</td>
</tr>
<tr>
<td>- - volume of</td>
<td>27</td>
</tr>
<tr>
<td>Atomic beam resonance <em>(Rabi's method)</em></td>
<td>375ff.</td>
</tr>
<tr>
<td>Atomic clock</td>
<td>382</td>
</tr>
<tr>
<td>Atomic model</td>
<td></td>
</tr>
<tr>
<td>- of Bohr</td>
<td>55</td>
</tr>
<tr>
<td>- of Prout</td>
<td>2</td>
</tr>
<tr>
<td>- of Rutherford</td>
<td>3, 40, 100</td>
</tr>
<tr>
<td>Atomic nucleus</td>
<td></td>
</tr>
<tr>
<td>- combined motion of</td>
<td>361</td>
</tr>
<tr>
<td>- general</td>
<td>27, 37ff., 47, 361ff.</td>
</tr>
<tr>
<td>- magnetic moment of</td>
<td>362ff.</td>
</tr>
<tr>
<td>- paramagnetism of</td>
<td>207</td>
</tr>
<tr>
<td>- spin</td>
<td>362f.</td>
</tr>
<tr>
<td>Atomic number <em>(nuclear Charge number)</em></td>
<td>27ff., 46</td>
</tr>
<tr>
<td>Atomic optics</td>
<td>79</td>
</tr>
<tr>
<td>Atomic spectra, influence of the nucleus</td>
<td>361ff.</td>
</tr>
<tr>
<td>Atomic weight</td>
<td></td>
</tr>
<tr>
<td>- determination of</td>
<td>5</td>
</tr>
<tr>
<td>- general</td>
<td>5, 29, 34</td>
</tr>
<tr>
<td>Atomism</td>
<td></td>
</tr>
<tr>
<td>- general</td>
<td>1</td>
</tr>
<tr>
<td>- of electricity</td>
<td>2</td>
</tr>
<tr>
<td>- of energy</td>
<td>2</td>
</tr>
<tr>
<td>- of heat</td>
<td>2</td>
</tr>
<tr>
<td>Attractor</td>
<td>252</td>
</tr>
<tr>
<td>Auger effect</td>
<td>332f.</td>
</tr>
<tr>
<td>Auger electron</td>
<td>333</td>
</tr>
<tr>
<td>Avogadro's hypothesis</td>
<td>2, 5</td>
</tr>
<tr>
<td>Avogadro's number</td>
<td>6</td>
</tr>
<tr>
<td>Backscattered electrons</td>
<td>63</td>
</tr>
<tr>
<td>Balmer formula</td>
<td>3, 98, 105</td>
</tr>
<tr>
<td>Balmer series</td>
<td>98ff., 108, 114f., 195, 324</td>
</tr>
<tr>
<td>Band spectra</td>
<td>95</td>
</tr>
<tr>
<td>Beat frequency</td>
<td>401</td>
</tr>
<tr>
<td>Beer's law</td>
<td>13</td>
</tr>
<tr>
<td>Bell's inequalities</td>
<td>408ff.</td>
</tr>
<tr>
<td>Bell's inequalities, experiments</td>
<td>411</td>
</tr>
<tr>
<td>Benzene</td>
<td>458</td>
</tr>
<tr>
<td>Bergmann series</td>
<td>177</td>
</tr>
<tr>
<td>Binary system</td>
<td>413</td>
</tr>
<tr>
<td>Binding energy</td>
<td>101, 115, 313, 328, 334, 405, 440, 454</td>
</tr>
<tr>
<td>- of inner electrons</td>
<td>328</td>
</tr>
<tr>
<td>Biot-Savart, law of</td>
<td>193</td>
</tr>
</tbody>
</table>
Bit 413
Black body radiator 51
→, energy density of 51
Bloch equations 241ff.
Bohr-Sommerfeld theory 115f.
Bohr magneton 184, 225, 362
Bohr model
→ for hydrogen-like Systems 104f.
→ for the hydrogen atom 95ff.
Bohr postulates 100f.
Bohr radius 104
Bohr's atomic model 55, 95
→, spin-orbit Splitting in 192ff.
Bohr's correspondence principle 117f., 213
Bolometer 208
Boltzmann constant 7
Boltzmann distribution 8, 10, 56, 309, 432
Bose-Einstein condensate 433
Bose-Einstein condensation 432, 434
Bose-Einstein condensation, quantum theory 435
Bose-Einstein statistics, distribution function 433
Boundary conditions for eigenvalue equations
131, 138
Boyle's law 10
Brackett series 99
Bragg reflections, law of 17
Bragg's rotating crystal method 15f., 17
Breit-Rabi formula 374
Bremsstrahlung 50
Brownian motion 7
Cadmium atom, normal Zeeman effect 209, 213
Carbon 456
Carbon atom
→, Grotrian diagram 317
→, valence of 345
Cathode rays 23ff., 37
Cavity radiation 2, 51ff.
Central field
→, m degeneracy 167
→, radial part of the wavefunction 161ff.
Centrifugal acceleration 30f.
Centrifugal force 30, 100f., 211
Chaos 251
→ in classical physics 251
→ in quantum mechanics 254
→, resonances in 254
Charge cloud 174
Chemical bonding 439ff.
→, heteropolar 439
→, homopolar 439
Chemical shift 335, 379
Clauser-Horne-Shimony-Holt inequality 408, 411
Cloud chamber 39, 41
Coherence length 394
Coherence of laser light
→, spatial 387
→, temporal 387
Coherent light 273
Coherent spin motion 242f.
Coincidence circuit 63
Collective measurement 431
Collective radiation 388
Collisional broadening 309
Collision radius 38
Colour centres 208
Commutation relations
→ and simultaneous measurability 139ff., 153, 234
→ for angular momentum operators 141, 157
→ for creation and annihilation operators in the harmonic oscillator 144
Compton effect 60f., 62f.
Compton-scattered radiation 60
Compton shift 62
Compton wavelength 62f.
Conduction electrons 208
Contact interaction 366
Correlation functions 409
Correspondence principle 117, 213
Coulomb field 41
Coulomb force 101f., 211
Coulomb gauge 279
Coulomb interaction energy 350, 355
Coulomb potential 41, 47, 153
→, l degeneracy 167
→, radial part of the wavefunctions 163ff.
Coulomb repulsion 41
Coulomb scattering 39f.
Counter-field method 58
Covolume 15, 18
Creation operator
→ for spherical harmonics 157
→ for the harmonic oscillator 145, 279
d series 176f.
D term 177, 315
Dalton's law 5
Davisson-Germer experiment 74f.
de Broglie hypothesis 75
de Broglie relations 81, 126
de Broglie wavelength 79
de Broglie waves 75, 84, 125, 129
Debye-Scherrer method 18, 77
Decay time, atomic 308
Decoherence 405ff., 416
Decoupling of spin and orbital angular momenta 215f.
Degeneracy 116
→ and perturbation theory 267
→, l degeneracy in hydrogen 167, 269
→, m degeneracy with spherical symmetry 167
Degenerate wavefunctions
→, orthogonality 441f.
→, transformation behaviour 298
Delta function 87, 461ff.
Density of states 8
Deuterium 106
Diamagnetism 186
Diffraction
- of an electromagnetic wave 49
- of an electron beam by a slit 77, 85f., 88
Digital Computers 413, 432
Dipole, oscillating 305ff.
Dirac equation 117, 243ff., 246
Dirac sea 248f.
Dirac theory 197f.
Directional degeneracy 212
Directional focussing 32
- Dempster method 32
Directional quantisation 185f., 189f., 192, 205
- of a magnetic moment 181
Distribution functions 432
Diving orbits 116, 174
Doppler broadening 195, 199, 310, 402, 404
Doppler-free saturation spectroscopy 402ff.
Doppler-free spectroscopy 200
Doppler-free two-photon absorption 404f.
Doppler shift 309, 402
Double focussing 32
Double resonance 218ff., 310
Double-slit experiment 404
Doublet splitting 192, 195
Doublet state 317
Doublet structure 181, 191f.
Dressed states 409
Dualism, wave-particle 63f., 86f.
Dye lasers 388
Eccentricity of elliptical orbits 115
Ehrenfest’s adiabatic invariance principle 339
Eigenfunction
- general 138
- of angular momentum 155ff.
- of the harmonic oscillator 147
- of the particle in a box 125ff.
- see also Wavefunctions
Eigenvalues
- general 138
- see also Energy (eigenvalues)
Einstein coefficients 56
Einstein-de Haas effect 181
Einstein-de Haas experiment 188f.
Einstein-Podolsky-Rosen (EPR) paradox 407ff.
Electric dipole moment 261
Electric potential 223
Electrolysis 7, 36
Electromagnetic spectrum 95
Electron 2, 29, 46, 62, 67, 109
- angular momentum of 205
- bare 284f.
- charge of 70
- collisions 111f., 114
E = mc^2 6, 57
Energy (eigenvalues)
- of the harmonic oscillator 146f.
- of the hydrogen atom 165
- of the particle in a box 126f.
Energy filter 30
Energy level 100ff.
- statistics in chaos 254, 255
Energy-mass equivalence 6, 57
Energy quantisation 53
Energy shift, relativistic 116
Energy spectrum
- of the Dirac equation 247
- of the Klein-Gordon equation 247
Energy (Splitting)
- of an electron in an electric field 261f.
- of a spin in a magnetic field 228f.
- of orbital angular momentum in a magnetic field 225
- see also Interaction energy
Energy term diagram 100
Energy terms, notation 196
Entangled states 409
Entanglement 406ff.
Equivalence principle 73
Error correction 416
ESCA 334f.
ESR apparatus 207f.
ESR spectroscopy 205, 374
Ethene 458
Exchange integral 442, 452
Exchange term in the Hartree-Fock method 357
Excited states 347f.
Exclusive or 414
Exotic atoms 120ff.
Expectation value 133ff.
–, see also Observation value, mean value
Extinction coefficient 330
Extremal principle, equivalence to the Schrödinger equation 357f.

f series 177
Fabry-Perot interferometer 209, 399f.
Faraday constant 7, 70
Fermi contact interaction 366
Fermi-Dirac statistics, distribution function 432
Feynman diagram 283
Field emission microscope 21
Field ionisation 118
Field quantization 276ff.
Fine structure 181ff., 191ff., 195, 311, 361
– constant 116
– interaction 197, 364
– of the hydrogen atom 197
– of x-ray absorption edges 332
– of x-ray spectra 328
– splitting 195, 217, 312, 326, 406
Fixed point 251
Fluctuating forces in the laser 394
Fluctuations of phases 406
Fourier transformation 308, 378, 381
Fowler series 107
Fractional distillation 35
Franck-Hertz experiment 112f.
Fraunhofer lines 103
Frequency of an electromagnetic wave 96
Fresnel biprism 76
Fresnel diffraction 76

Galton board 404
g factor 184, 208, 320, 363
– for orbital magnetism 217
– of the electron 187f.
– of spin magnetism 217
–, precision measurement 369f.
g series 177
Gamma radiation 95, 111
Gas centrifuge 35
Gas constant, universal 7
Gas discharge 23f., 29, 37
Gas discharge tube 23
Gas laser 402ff.
Gaussian line 309
Giant atoms 117
Grating spectrometer 209
Greenberger-Horne-Zeilinger state 412

Gross-Pitaewskii equation 435
Grotrian diagram 175f., 317
– of the carbon atom 317
– of the lithium atom 175
– of the mercury atom 319
– of the nitrogen atom 317
– of the potassium atom 178
– of the sodium atom 176
Ground state terms 342ff.
Group velocity 84
Gyromagnetic ratio 181, 185, 188ff., 362, 364
– of nuclei 207
– of the electron 208

Half-intensity thickness 330
Hamilton function for an electron in an electromagnetic field 223
Hamilton operator
–, general 130, 465
– for an electron in an electromagnetic field 224
– for the many-electron problem 349ff.
Hanle effect 406ff.
Hartree-Fock method 338, 349ff.
Hartree method 354ff.
Heat baths 406
Heavy-ion accelerator 27, 367, 370
Heavy ions 109
Heisenberg uncertainty relation 87ff.
Heitler-London method 450, 454
Helium atom 311
–, Lamb shift 200
–, spectrum 311f.
–, term diagram 312
Helium ion 313
Helium-neon laser 389
h/e measurement 59
Hermite polynomials 147
Hertzian dipole 104, 212, 305f., 325
Hertzian vector 305
Hidden variable 407, 412
Hidden-variable hypothesis 408ff.
High-frequency mass spectrometer 34
Hilbert space 414
Hund-Mulliken-Bloch method 455f.
Hund’s rules 345ff.
Huygen’s principle 17
Hybridisation 345, 456ff.
Hydrogen atom
–, Bohr’s model 95ff.
–, eigenvalues 165
–, fine structure 197
–, hyperfine structure 368ff.
–, in ESR 373
–, in high magnetic fields 249ff.
–, in low quantum states 256ff.
–, degeneracy 167
–, Lamb shift 198ff., 369
–, radial part of the wavefunctions 163ff., 166
–, shell structure 338f.
Isotope shift 106
Isotope composition 32, 34
Isotopy 27, 33

$jj$ coupling 318ff.
Jordan's rule 137, 224

$K$ edge 327
$K$ series 326ff.
$K$ shell 178, 330
Kepler's laws 115f.
Kinetic energy 101
Kinetic theory of gases 2, 10
Kirchhoff's law 57
Klein-Gordon equation 245ff.
Klystron 198, 208

$L$ series 326ff.
$L$ shell 178, 333
$l$ degeneracy, lifting of 171ff.
Laguerre polynomials 165
Lamb dip 403
Lamb shift 198ff., 248, 276ff., 369, 404
Landau wave function 250
- levels 250
Landé factor 184, 213, 216ff., 236, 320
Laplace operator 130
- in spherical polar coordinates 154
Larmor frequency 185, 207, 211
- of the nuclear spin 375
Laser amplitude, complex 394ff.
Laser conditions 391
Laser cooling 408ff.
Laser equation 395
Laser light 118, 387
Lasers 387ff.
- in the photon picture 388, 390ff.
- in the wave picture 388, 393ff.
- stationary states 392
Laser threshold 393
Lattice constant 8, 16
Lattice, face-centred cubic 8
Lattice planes 16
Laue method 16f., 78
Law of constant and multiple proportions 2
LCAO method 456, 459
Lead, x-ray absorption cross section 332
Legendre functions, associated 155
Legendre polynomials 155
- plot of the simplest 156
Lenard's cathode ray tube 37
Lenard window 37
Letokhov, method of 403
Level crossing 406ff.
Level-crossing spectroscopy 406ff.
Lifetime 306, 311, 406, 408
Lifting of $l$ degeneracy 171ff.
Light as a particle 50
Light as a wave 49
Light emission, exponential decay in spontaneous 
306ff.
Light pulse, ultra short 387
Light quantum 50, 54, 57, 281ff.
\(\rightarrow\), angular momentum of 213, 305
Light quantum hypothesis 57f, 62
Limit cycle 251
Linear combination
\(\rightarrow\), antisymmetric 351
\(\rightarrow\), of eigenfunctions belonging to the same energy 267, 441
\(\rightarrow\), see also Superposition
Line spectra 95
Linewidth of a laser 387, 396
\(\rightarrow\), homogeneous 274, 309
\(\rightarrow\), inhomogeneous 309
\(\rightarrow\), natural 308, 402ff, 406
Lithium atom
\(\rightarrow\), shell structure 338f, 344
\(\rightarrow\), spin-orbit splitting in 217
\(\rightarrow\), term diagram 175
Local hidden variable 411
Local realistic theory 408
Logical element 413
Logical negation 416
Lorentz equation 73
Lorentz force 30, 211, 223
Lorentz formula 73
Lorentz invariance 73
Lorentz line shape 308
Loschmidt's number (Avogadro's number) 6f, 70
Losses, in the laser process 390, 394
LS coupling 314ff, 318f.
Lyapounov exponent 255
Lyman series 99f, 106, 108f, 324
\(\rightarrow\), muonic transitions in 111

\(M\) series 326
\(M\) shell 178
Magnetic dipole moment 182ff, 362
\(\rightarrow\), nuclear 362
\(\rightarrow\), measurement of 375ff.
\(\rightarrow\), sign of nuclear 368
Magnetic dipole, potential energy 182
Magnetic dipole radiation 304
\(\rightarrow\), field strength of 183
\(\rightarrow\), induction of 183
Magnetic dipole transitions 206
Magnetic moment 205, 214, 311
\(\rightarrow\), directional quantisation of 181ff.
\(\rightarrow\), measurement of 189ff.
\(\rightarrow\), of an electron 186ff.
\(\rightarrow\), of a spin 229
\(\rightarrow\), of atomic nuclei 362ff.
\(\rightarrow\), of many-electron atoms 320
\(\rightarrow\), of the orbital angular momentum 182ff, 225
\(\rightarrow\), with spin-orbit coupling 215ff.
Magnetic moments
\(\rightarrow\), of inner electrons 191
\(\rightarrow\), quantum number 155, 167, 186, 205, 338ff.
\(\rightarrow\), spin quantum number 338
Magnetic resonance tomography 379ff.
Magnetisation, macroscopic 181
Magnetomechanical parallelism 184
Many-electron atoms 311
Many-electron problem 173, 347ff.
Maser, one atom 23
Mass-and-spring oscillator 54
Mass, bare 284ff.
Mass change, relativistic 73, 115f.
Mass defect 33
Mass density 8
Mass number 5, 29, 33
Mass of the atom 5
Mass of the photon 57
Mass ratio proton/electron 106
Mass separation by diffusion 35
Mass spectrometer 32f.
\(\rightarrow\), applications 34
Mass spectrometry 29, 32, 34
Mass unit 5
Matrix element
\(\rightarrow\), of the perturbation operator 265
\(\rightarrow\), of the spin 227f.
\(\rightarrow\), optical 268, 271, 295, 300ff.
Matter waves 3, 74
\(\rightarrow\), see also de Broglie waves
Maxwellian collision cylinder 10
Maxwell's equations 49, 305, 305
Maxwell's theory 49
Maxwellian velocity distribution 11, 309, 402f.
Mean free path 14
Mean value
\(\rightarrow\), general 133
\(\rightarrow\), of the angular momentum 137
\(\rightarrow\), of the energy 136
\(\rightarrow\), of the momentum 134f.
\(\rightarrow\), of the position 133f.
\(\rightarrow\), of the spin 230ff.
Measurement, simultaneous 139, 153, 234
Measurement value = eigenvalue 133, 139
\(\rightarrow\), expectation value (mean value) 133, 139
Mercury atom
\(\rightarrow\), double resonance in 219
\(\rightarrow\), Grotrian diagram for 319
Mercury low-pressure lamp, spectrum of 319
Metastable 311
Methane 456f.
Microwave resonator 208
Microwave spectroscopy 195
Microwaves 95, 118, 205
Miller indices 16
Millikan experiment 70
Molecular beam resonance (Rabi method) 375f.
Molecular radicals 208
Molecular weight determination 107
Mole mass (molecular weight) 8
Momentum eigenfunctions 138
Momentum eigenvalues 138
Momentum filter 31
Momentum operator 137f., 224
Momentum uncertainty 89ff.
Moseley diagram 327
Motion of the nucleus 105ff.
Multiple excitations 321
Multiplet structure 191f.
- of hydrogen 114
- of the Balmer series 114f.
Multiplicity 196, 311, 314, 317
Multipole radiation 305ff.
Muon 109, 120
Muon neutrino (neutretto) 109
Muonic atom 109ff.
- orbital radius 109
Muonium 120ff.
Natural linewidth 402ff.
Negation 414
Neutrino 109
Neutron 7, 29, 46, 109f.
Neutron star 256, 257
Nitrogen atom, Grotrian diagram 317f.
NMR see Nuclear magnetic resonance
Noble gas atoms 27
Noble gas configuration 171, 177, 340
Noble gases 340
Noise effect in the laser 391, 393
Nomenclature for terms 316f.
Nondestructive single-photon detection 413ff.
Normalisation of the wavefunctions
- for the harmonic oscillator 146
- for the particle in a box 87, 128
- general 87
- in infinite space 87, 464f.
Normalisation of the Laguerre polynomials 165
Normalisation of the spherical harmonics 159ff.
Normalisation of the spin wavefunctions 228, 230
Nuclear angular momentum 362
Nuclear charge 5, 27, 29, 41, 45f., 326, 361
- effective 330
Nuclear charge distribution 383
Nuclear density 46
Nuclear force 46
Nuclear g factor 363, 376
Nuclear magnetic resonance 207, 364, 376ff.
- apparatus for 377
Nuclear magneton 229, 362
Nuclear motion 105
Nuclear photoeffect 60
Nuclear potential 47, 173
- effective 173f.
Nuclear quadrupole moment 383ff.
Nuclear radius 41, 46f.
Nuclear spin 362, 412
- Larmor frequency 375f.
- measurement of 375ff.
Nuclear structure 46
Nuclear structure of the atom 37
Nucleon 7
Nucleus 39
- gyromagnetic ratio 376
- magnetic moment 362
- measurement of 375ff.
Nuclide 31
Occupation numbers of atomic states 273, 275
- in the laser 390ff., 402
Occupation probability (density) 85ff., 167f., 444ff.
ODMR 220
One-electron states 171
Operator
- for angular momentum 137f.
- in spherical polar coordinates 159
- for the energy 130f.
- for the momentum 137
- for the position 137
- spin 226ff.
Operators
- commuting 140
- general 137
- Hermitian 441, 465f.
- linear 441
Optical detection of ESR 220
Optical molasses 409
Optical pumping 212, 218ff.
- in the laser 388, 391
Orbital 115f.
Orbital angular momentum 103, 115, 181f., 186, 196, 311, 364
- quantisation of 103
Orbital angular momentum quantum number 115, 196, 338f.
Orbital degeneracy 115f., 181, 196
- lifting of 115f., 171, 313, 329
Orbital magnetism 181, 188, 213
Orbital magnetic moment 213
Orbital radius 102f.
Orthogonality of wavefunctions 228, 265, 442
Orthohehium 311
Oscillator
- electromagnetic 53
- quantum mechanical 142ff.
- zero point energy 54, 91, 146
Overlap integral 450
p series 177
P term 177, 315
Palladium 344
Parabola method of Thomson 29ff.
Parahelium 311
Parallel-plate interferometer 399
Paramagnetic ions 208
Paramagnetic states 208
Paramagnetism 186, 206, 344
Parity 297
Partition function 8
Paschen-Back effect 210, 217ff., 320
- in hyperfine structure 370f.
Paschen series 99f.
Paul trap 24
Pauli equation 232f.
Pauli principle 171, 313f., 316, 338, 345, 347, 448, 450
Penetration depth of particles 39
Perihelion, rotation of 116
Periodic system of the elements 2, 27ff., 337ff., 342f.
Permeability constant of vacuum 184
Permeability, magnetic 320
Perturbation theory
- with degeneracy 267ff.
- without degeneracy 264ff.
Pfund series 99f.
Phase diffusion in the laser 396
Phase velocity 84
Photocell 60
Photoelectric effect 58, 64, 334
- inner 60, 332f.
- internal 60
- nuclear 60
- outer 60
Photoelectric effect 58
Photoelectron spectroscopy 334ff.
Photomultiplier 23f.
Photon 49f., 51, 54, 62, 281ff.
- mass of 57
- spin of 213, 305
Photon echo 273ff.
Photon occupation number in the laser 390
π bonding 458
Pickering series 108
Pions 109
π pulse, π/2 pulse 240, 273f.
π transition 302
Planck’s constant 53f., 88
- precision measurement of 325
Planck’s quantum of action 53f., 88
Planck’s radiation formula 53, 57
- derivation by Einstein 54
- transition probability per unit time 390
Platinum 344
- fine structure of the x-ray spectra 329
- x-ray absorption 331
- x-ray emission 331
Poincaré cross section 252, 253
Polarisability 262
Polarisation
- as a quantum mechanical source term 305ff.
- Zeeman effect 210, 212, 213, 218ff.
Quantum electrodynamics (QED) 187, 199, 201, 248, 276ff.
Quantum "exclusive or" 416
Quantum field theory 245, 249
Quantum information 431
Quantum information theory 432
Quantum jumps 111ff.
Quantum number of the total angular momentum 191ff.
Quantum numbers 155, 164, 313, 338
Quantum theory 3
  – of chemical bonding 439ff.
Quantum yield 332
Quarks 70
Quartet state 317
Quasi-periodic motion 252
Quibits (quantum bits) 414, 415
Quintet state 317
Rabi atomic or molecular beam resonance 375ff.
Rabi frequency 409
Radial part of the wavefunction
  – in a central field 161
  – in the hydrogen atom 163ff.
Radiation density 56
Radiation flux density 53
Radiationless processes 332
Radioactive decay 9, 22
Radio astronomy 99
Radio frequencies 95, 207
Ramsauer effect 74
Rare earths 27, 344
Rate equations for the laser 390ff.
Rayleigh-Jeans radiation law 53, 57
Rayleigh scattered radiation 60
Recombination
  – optical 389, 390
  – radiationless 392
Reduced mass 105
Relativistic correction 197
Relativistic quantum theory 187, 243ff.
Relativity, theory of 73
Relaxation processes 219
Relaxation time of a spin
  – longitudinal 243
  – transverse 243, 274
Renormalisation 201, 281ff., 285
Renormalisation theory 201, 281ff., 285
Residual gas analysis 34
Resolution, spectral, of closely spaced levels 310, 400ff.
Resolving power
  – of a mass spectrometer 32ff.
  – of a microscope 20
  – spectral 113, 310, 361, 400ff., 411
Resonance, covalent-ionic 454ff.
Resonance fluorescence 96
Resonance light 406
Resonance line 177
Resonance
  – of the magnetic field with a spin frequency 236, 238
  – of the radiation field with electron transitions 271ff.
Resonances in chaos 254
Resonant cavity 413ff.
Resonator 208
Rest energy of the electron 63
Rhodium 344
Ritz combination principle 99
Rotating wave approximation 272
Russell-Saunders coupling 314ff.
Russell-Saunders terms 348
Rutherford scattering 39ff., 41
  – anomalous 40, 46
  – experimental results 46
Rutherford scattering formula 41ff., 45
Rutherford's atomic model 3, 40, 46, 100
Rydberg atoms 117ff.
Rydberg constant 98, 105, 361
Rydberg formula 99
Rydberg states 118
s series 177
S term 177, 315
Saturation spectroscopy, Doppler-free 402ff.
Scalar product of spin wavefunctions 227, 231
Scanning electron microscope 21
Scanning tunnel microscope 21
Scattering of wave functions 254, 255
Scattering coefficient 12ff., 38, 74
Scattering cross section 11
Scattering formula of Rutherford 45
Scattering of particles 39
Schrödinger's cat 405
Schrödinger equation
  – for a free particle 129
  – for a spin in a magnetic field 228ff.
  – for a two-level atom in a radiation field 270ff.
  – for the harmonic oscillator 142
  – for the many-electron problem 349ff., 354ff.
  – for the radial part 155, 161
  – general 129ff.
  – of the electron with spin-orbit coupling 233ff.
  – time independent 131
  – with a central field 154
  – with an electromagnetic field 225
Scintillation detector 23ff., 40, 111
Screening effect 173
Screening number 327
Secondary electron multiplier (electron multiplier) 23ff.
Secondary series 176
Secular determinant 267
Sedimentation equilibrium 7
Selection rules and symmetry 300ff.
  – for collisional excitation 114
Subject Index

Selection
- for optical transitions 174, 192, 197, 213, 215, 218, 225, 305, 315ff., 329, 349, 371
- in multipole radiation 305ff.
Self energy 201
Semi-conductor detector 111
Separation ansatz for wavefunctions in a central field 154
Separation tube 35
Series limit 98, 327, 330
Series-limit continuum 99, 331
 Sextet state 317
Shells, inner 173, 179, 323ff., 334
Shell structure 171ff., 341
- of atoms 337ff.
- of the atomic energy levels 341
 σ transition 302
Silver, Auger effect 333
Single-mode laser 390
Singlet state 316
Singlet System 311f.
Sodium atom 344
- experiment showing the Zeeman effect 209
- hyperfine structure of 368
- resonance fluorescence 96
- shell structure of 338f.
- spectral series of 176
Sodium
- term diagram 176
Sodium D line 176f., 181, 195, 329
- anomalous Zeeman effect in 209, 214f.
- hyperfine splitting of 369
- optical pumping 219
- spin-orbit splitting 217f.
- Stark effect 262
Sommerfeld-Kossel displacement theorem 109
Sommerfeld fine structure constant 116, 198
Sommerfeld model 174
Sommerfeld’s extension of the Bohr model 114f.
Sommerfeld’s second quantum number 115
Space, atoms in 118
Space lattice interference 16
Spectral analysis 97
Spectral energy density 51
Spectral regions 95
Spectra of the alkali atoms 171ff.
- continuous 95
- hydrogen-like 107
- optical 3
Spectrograph 361
Spectrum of the hydrogen atom 97ff.
Spherical harmonic functions 160
- the simplest 161
Spin 158, 182ff., 186ff., 205, 226ff., 312, 362f., 407, 448ff., 455
- of atomic nuclei 362ff.
Spin expectation value 230ff.
Spin flip 236, 239ff., 273f., 315f.
Spin magnetism 181ff., 188, 214f.
Spin matrices 226ff., 246
Spin motion, (in)coherent 242
Spin operators 226ff.
Spin-orbit coupling 191, 215ff., 232ff., 312, 314, 318
Spin-orbit coupling constant 194
Spin-orbit coupling energy 194, 197
Spin-orbit interaction 314, 328, 345, 365
Spin-orbit interaction energy 194, 233
Spin-orbit splitting 196
- in Bohr’s atomic model 192ff.
Spin precession 193, 230ff.
Spin quantum number 187, 196, 226, 338
Spin resonance technique 240
Spin thermometer 22f.
Spin variable 350
Spin wavefunctions 226ff.
Spontaneous emission in the laser 388, 390, 394
Square potential (particle in a box) 125ff.
Stadium billiard 255
Stark effect 261f.
- linear 261, 267ff.
- quadratic 261, 263ff.
State, bound 444
States, non-bound 444
Statistical interpretation of quantum mechanics 85ff.
Stefan-Boltzmann law 52f.
Stern-Gerlach experiment 181, 187f., 190ff., 375
Sternheimer factor 385
Stimulated emission in the laser 387, 390
Superposition
- of spin wavefunctions 227, 230, 237
- of wavefunctions for different atomic levels 304, 401
- principle 403
- see also Linear combination
s-wave scattering length 435
Symmetries and selection rules 295ff.
Symmetry transformation 296ff.
Synchrotron radiation 49f.
Teleportation 432
Temperature determination, spectroscopic 104
Term analysis 215, 218
Term diagram, complete 347ff.
Terms, nomenclature 316f.
Thermal conductivity 15
Thermal diffusion 35
Thermal radiation 50f.
Thermonuclear emission 111
Thomas factor 193
Thomson’s parabola method 72
Time-of-flight mass spectrometer 34
Total angular momentum 314, 316, 364
Total angular momentum quantum number 191f., 196, 217
Total orbital angular momentum 191f., 194, 212, 217, 314, 317
Total spin 314
Total wavefunction 346
Trajectory 251
Transformation behaviour of wavefunctions
  – on reflection 296ff.
  – on rotation 298ff.
Transition elements 344
Transitions
  – allowed and forbidden 113f.
  – see also Selection rules
Transuranic elements 27
Triplet state 316f.
Triplet system 311f.
Tunnel effect 445ff.
Turbine drive, for gyroscope 206
Two-electron system 349
Two-level system
  – atomic 270, 273ff.
  – spin 226, 273ff.
Two-photon absorption, Doppler-free 404f.

Ultraviolet catastrophe 53
Uncertainty relation
  – for energy and time 89
  – Heisenberg’s 87ff.
Unit cell 8
Uranium isotopes 35

Valence, (electro-)chemical 338, 344
Valence electron 171, 314, 344, 456
Valence electron (radiant electron) 117
  – screening 173
Van der Waals equation for real gases 15
Variational principle, equivalence to the
  Schrödinger equation 357ff., 447
Vector model for hyperfine structure 364
Vector model for spin-orbit coupling 191ff.
  – quantum mechanical basis 232ff.
Vector potential 223, 304
Velocity dependence of mass 73
Velocity focussing (Aston’s method) 32
Virtual processes 201
  – transition 408
Viscosity 14, 18
Volume effect 361
Volume of an atom 8

Wannier exciton 123
Wave character of light 49f.
Wave, electromagnetic 49

Wave functions, radial part
  – in a central field 161ff.
  – in a Coulomb field 163ff.
  – see also Eigenfunctions
Wavelength standard 96
Wavelength, unit of 96
Wavenumber 97
Wave packet 81ff.
  – flowing apart of 84
White dwarf 256, 257
Wien’s displacement law 53
Wigner distribution 256
Wigner function 255
Wigner-Weisskopf theory 401
Work function 60, 325

X band 208
X-radiation 50, 60, 95
  – characteristic 326ff.
  – from outer shells 323ff.
  – wavelength 95, 323ff.
  – measurement of 9
X-ray absorption coefficient 332
X-ray absorption edges 327
X-ray absorption spectra 330ff.
X-ray bremsstrahlung spectrum 324ff.
X-ray detectors 323
X-ray diffraction by crystals 8, 15
  – Laue method 15
X-ray fluorescence 327
X-ray line spectrum 326ff.
X-ray quantum 323ff.
X-ray spectra 323ff.
  – characteristic 324
  – fine structure in 328ff.
  – white 325

Zeeman components 210f.
  – polarisation of 212, 218ff.
Zeeman effect 208ff., 261
  – anomalous 186, 209f., 214f., 232ff., 320
  – classical electron theory of 210f.
  – in hyperfine structure 370ff.
  – ordinary 209f., 212ff., 218, 320
  – vector diagram 370
  – vector model 212ff.
Zeeman energy
  – of a proton 373
  – of an electron 373
  – of the nucleus 373
Zeeman spectroscopy 205
Zeeman triplet, ordinary 213
Zero point energy of the harmonic oscillator 91, 146
Zero point oscillations of atoms 92
### Energy Conversion Table

<table>
<thead>
<tr>
<th></th>
<th>J</th>
<th>eV</th>
<th>cm⁻¹</th>
<th>K</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Joule (J)</td>
<td>1</td>
<td>6.24146 × 10¹⁸</td>
<td>5.03404 × 10²²</td>
<td>7.24290 × 10²²</td>
</tr>
<tr>
<td>1 eVolt (eV)</td>
<td>1.60219 × 10⁻¹⁹</td>
<td>1</td>
<td>8.06548 × 10³</td>
<td>1.16045 × 10⁴</td>
</tr>
<tr>
<td>1 cm⁻¹</td>
<td>1.98648 × 10⁻²³</td>
<td>1.23985 × 10⁻⁴</td>
<td>1</td>
<td>1.43879</td>
</tr>
<tr>
<td>1 K</td>
<td>1.38066 × 10⁻²³</td>
<td>8.61735 × 10⁻⁵</td>
<td>6.95030 × 10⁻¹</td>
<td>1</td>
</tr>
</tbody>
</table>

### Explanation

The energy $E$ is quoted in Joule (J) or watt-seconds (Ws)

$$ 1 \text{ J} = 1 \text{ Ws} . $$

In spectroscopy, one frequently quotes the term values in wavenumbers $\tilde{\nu} = E/hc$.

The conversion factor is

$$ E/\tilde{\nu} = hc = 1.98648 \times 10^{-23} \text{ J/cm}^{-1} . $$

Another energy unit, especially in collision experiments, is the electron volt (eVolt, eV). The voltage $V$ is given in volts, and the energy conversion factor is obtained from $E = eV$:

$$ E/V = e = 1.60219 \times 10^{-19} \text{ J/V} . $$

In the case of thermal excitation with the heat energy $kT$, the absolute temperature is a measure of the energy. From $E = kT$ we obtain the conversion factor

$$ E/T = k = 1.38066 \times 10^{-23} \text{ J/K} . $$