Understanding the differences in neutrino and charged-fermion flavour structures

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We present a new mechanism to explain naturally and through a common flavour symmetry the mildly hierarchical neutrino masses with large mixings and the hierarchical Yukawa matrices with small mixing angles. Although this mechanism is not linked to any particular flavour symmetry, it is particularly simple in the framework of a SU(3) flavour symmetry. In this model, we obtain exactly maximal atmospheric mixing, large although not maximal solar mixing, and a normal neutrino hierarchy. All neutrino parameters are basically fixed. For instance, we predict that the angle $\theta_{13}$ comes entirely from the charged lepton sector and, using the difference of the solar angle from maximality, we can fix the mass of the lightest neutrino.

In the last decade, oscillation experiments have confirmed that neutrinos have non-vanishing mass and they do mix. However, it is not easy to understand the measured values of neutrino masses and mixing angles. These parameters, specially when compared with quarks, are extremely unusual. On one side the quarks have masses of the order of the electroweak scale, a strong intergenerational hierarchy and small mixing angles. On the contrary, neutrino masses are much smaller than the electroweak scale, they are not very hierarchical and the mixing angles are large.

The seesaw mechanism can provide a very appealing explanation for the neutrino mass scale. But even within this mechanism, the intergenerational hierarchy and mixing angles remain a difficult problem. For example, in a SO(10) Grand Unified Theory (GUT) a full SM generation plus a right-handed neutrino are included in the $16$ representation. Therefore, up quark and neutrino Dirac Yukawa couplings are expected to be closely related. In these conditions, we could also expect a similar hierarchy of masses and small mixing angles in the neutrino sector. On the contrary, the experimental results force us to obtain near degeneracy and maximal mixing angles in the left-handed Majorana neutrino mass matrix from a strong hierarchy and small mixing angles in the Yukawa matrices. In the literature we can see a large variety of models that succeed to do this. However, these models are not completely satisfactory. In some cases, as we find for instance in $U(1)$ flavour models or in some SU(5) GUTs, large mixings come from the charged lepton sector, but here it is not straightforward to guarantee maximal atmospheric and large solar angles together with correct mass differences. In pure SO(10) or non-Abelian flavour symmetries, we usually have symmetric Dirac neutrino Yukawa matrices. Then, we have to obtain large mixings from small Cabibbo–Kobayashi–Maskawa (CKM) mixings. This is usually obtained at the cost of a certain tuning of the parameters. In any case, it is evident that the (flavour) symmetry structure of the low-energy effective neutrino mass matrix is radically different from the very hierarchical structure of charged leptons and quarks. This is the source of all our problems when trying to accommodate simultaneously quarks and leptons in the same flavour model. So far there is no convincing symmetry reason to ensure large neutrino mixings together with small quark mixing angles.

In this paper we present a very simple mechanism that allows us to obtain naturally correct neutrino masses and large mixing angles from, for instance, very hierarchical Yukawa matrices with small mixing. All this is obtained with all the fermion mass matrices including right-handed Majorana matrices sharing basically the same flavour structure. The only small differences are precisely due to the Majorana nature of the neutrino mass matrix. In the following, we briefly present the main ideas of this mechanism and then we will discuss some possible realization of this mechanism in flavour models.

In the literature, it is known that to obtain the experimental values of the neutrino mass differences through the seesaw mechanism, the hierarchy in the right-handed Majorana matrix must be approximately the square of the hierarchy in the Yukawas. The simplest possible realization of this idea would be to take $M_R = M_X (Y^{T} Y^{T})$. From this structure we would obtain exact neutrino degeneracy from any Dirac neutrino Yukawa matrices. Although this goes in the right direction, it is not enough: neutrinos are not degenerate, and large mixings are present in nature. However, continuing along the same lines, it is straightforward to obtain the required right-handed Majorana matrix from the seesaw formula itself:

$$M_R = v^2 Y^T \cdot (\chi L)^{-1} \cdot Y^{T} \cdot (\chi L)^{-1} \cdot Y^{T}$$

(1)

This equation suggests that the structure of the right-handed Majorana mass matrix is proportional to $Y^{T}$ on the left and the right, and that these matrices are connected by a third flavour matrix, which is simply the inverse of the low-energy effective neutrino mass matrix. This is the only possible solution once we fix the neutrino Yukawa matrices. This expression is analogous to the $M_R$ we would obtain in the two-step seesaw mechanism. Although Eq. (1) does not mean that we must go through a two-step seesaw to obtain $M_R$ with this structure, we think it is really worth trying to construct a right-handed Majorana matrix exactly along these lines.
Using this form it is easy to see that $M_R$ has the following features: i) The texture of $M_R$ is still diagonalized by similar rotations to the rotations diagonalizing the Yukawa matrices, and the hierarchy in $M_R$ is approximately the square of the hierarchy in $Y_\nu$. ii) Even in this situation, the structure of the left-handed neutrino Majorana matrix is **basically decoupled** from the texture of the Yukawa matrices of neutrinos and charged fermions. iii) The texture of the effective left-handed neutrino Majorana mass matrix is only determined by the structure of the connecting matrix $\chi_L^{-1}$. These features can explain the differences between neutrino and charged lepton textures and/or symmetries. In this way we can expect the textures $M_R$ and $Y_\nu$ to be determined by the same flavour symmetry.

It is very interesting to analyse what the required structure of this connecting matrix $\chi_L^{-1}$ is. Let us assume, that we have bimaximal neutrino mixings. The mixing matrix $U$ then is

$$U = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
\frac{1}{2} & -\frac{1}{2} & \sqrt{2} \\
-\frac{1}{2} & -\frac{1}{2} & \sqrt{2}
\end{pmatrix}.$$  \hspace{1cm}  (2)

Regarding the neutrino masses, we only know two mass differences but there is absolutely no reason to believe that one of the masses is exactly zero and thus we leave them free for the moment. Then, $\chi_L = U^* \cdot \text{Diag}(m_1, m_2, m_3) \cdot U^\dagger$, and trivially the inverse of this matrix is $\chi_L^{-1} = U^* \cdot \text{Diag}(\frac{1}{m_1}, \frac{1}{m_2}, \frac{1}{m_3}) \cdot U^\dagger$. Therefore,

$$\chi_L^{-1} = \frac{1}{m_3} \begin{pmatrix}
0 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} \\
0 & \frac{1}{2} & \frac{1}{2}
\end{pmatrix} + \frac{1}{m_2} \begin{pmatrix}
\frac{1}{2} & 1 & -\frac{1}{2} \\
\frac{1}{2} & 1 & -\frac{1}{2} \\
\frac{1}{2} & 1 & -\frac{1}{2}
\end{pmatrix} + \frac{1}{m_1} \begin{pmatrix}
\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2}
\end{pmatrix}.$$  \hspace{1cm}  (3)

This structure is indeed very simple and, given that all the elements in the matrices are of the same order, the only possibly small parameters that can order the different contributions are the neutrino mass eigenvalues themselves. Maximal atmospheric mixing is given by the last contribution in Eq. 3 and in fact this structure is **precisely** one of the main ingredients of $SU(3)$ flavour models. Although, in principle, it could be possible to reproduce this structure in other flavour models, we think it is a hint in favour of these models and we will exploit this relation in the following.

The main question now is whether it is possible in a complete model to obtain the structure given in Eq. 3 or, more exactly, in Eq. 3. To reach this goal we have to force the right-handed neutrinos to couple “only” through the combination $(\nu R)_i (Y_\nu^T)^j$. This combination carries exactly the same charges as $H \nu_L^\dagger$ if we assume that the Dirac Yukawa coupling is allowed. This means that it carries only one unit of lepton number plus the flavour charge that would be saturated by the left-handed neutrino. Now we have to remember that the Majorana matrix has still another fundamental feature: it violates lepton number (or $(B-L)$) by two units. This means that to obtain a Majorana matrix from a product of Yukawa matrices we would need to add a vacuum-expectation-value (vev) breaking lepton number.

We can think of a scalar field $\lambda$, with $L = -1$ and charged under the flavour symmetry saturating the quantum numbers of $(\nu_R)_j (Y_\nu^T)^j$, which gets a vev. In this way we would need a new mediator $S$, singlet under the SM, flavour and $(B-L)$ symmetries, with a relatively large mass to connect again with $(\nu_R^T)_i (Y_\nu^T)^j$. Clearly, this construction is allowed by all the symmetries. This would be a two-step seesaw mechanism for $((\nu_L)_i, (\nu_R)_j, S)$, and in fact it is simpler to discuss the physics in terms of the two-step seesaw structure. The $7 \times 7$ “neutrino” mass matrix here is

$$
\begin{pmatrix}
0 & Y_{ij} & 0 \\
0 & Y_{ij}^T & (\lambda)^T_k \cdot \nu_k \\
0 & (\lambda)^T_k \cdot \nu_k & M_S
\end{pmatrix} \cdot \begin{pmatrix}
(\nu_L)_i \\
(\nu_R)_j \\
S
\end{pmatrix}.$$  \hspace{1cm}  (4)

First, we notice that this structure can only give mass to one right-handed and one left-handed neutrino. The equivalent here of the connecting matrix in Eq. 3 would be $(\lambda \cdot \nu_R^T)^j$, which is singular, with only one non-vanishing eigenvalue. This can be solved by adding other fields (flavon vevs) coupling to $(\nu_R)_j (Y_\nu^T)^j$ in a similar way to generate masses to the other two neutrinos. In flavour theories we do have already other fields that can play this role; we present an example in a $SU(3)$ flavour theory below. Equation 4 is the structure we need, but clearly there are several conditions our model has to fulfil to obtain this matrix:

1. We must make sure that the same Yukawa matrix enters both the $(1,2)$ and $(2,3)$ blocks.
2. We should guarantee that there is no direct mixing between $\nu_L$ and $S$, i.e. the $(1,3)$ entry is vanishing or sufficiently small.
3. There could be other operators with a structure also allowed by our symmetries contributing directly to the $(2,2)$ block of Eq. 4, which should be sufficiently small.

The first condition is automatically satisfied if $\lambda$ behaves in the same way as $\nu_L$ under the flavour symmetries. Points 2 and 3 are very much related and solved simultaneously. If we look at point 2 in the context of a Froggatt–Nielsen mechanism, it means, taking into account $SU(2)_L$ and $(B-L)$ symmetries, that the coupling $\frac{1}{M_0} (\nu_L)_i (Y_\nu^T)^j \cdot \lambda_S H S,$ should be sufficiently small. Where $f^j$ is a Yukawa-like function of flavon vevs and $M_0$ one
of the flavour mediators that enter the Froggatt–Nielsen graph. Its contribution to left-handed neutrino masses would be 
\[ m'_\nu = \frac{v^2}{M_R} f^{ikj} (\lambda^i) f^{jli} (\lambda^l) \frac{v_{B-L}}{\mu^2 R}, \]
with \( \lambda = v_{B-L} \lambda \) and \( v \) the Higgs SU(2)-breaking vev. This has to be compared to the two-step seesaw contribution, which would be simply \( \frac{v^2 M_S}{v_{B-L}} (\lambda^i) (\lambda^j) \). Given that, at least some entries in the \( f_{ij} \) matrices can be of order 1, we do not consider any additional suppression from them. Then the condition that this contribution is small is simply
\[ \frac{v_{B-L}^2}{M^2} \ll \frac{M_S}{v_{B-L}^2} \Rightarrow \frac{v_{B-L}^4}{M^2} \ll 1. \]  
(5)

Point 3 gives a very similar condition. We have to notice that the difference between the structure in Eq. (4) and any other possible structure not proportional to the Yukawas is that they will not involve such completely neutral mediator fields. Notice that if \( S \) was the mediator field connecting the two \((B-L)\)-breaking vevs, both structures on the left and the right of \( S \) would necessarily be the Yukawas, as they are the leading couplings allowed by the flavour symmetries between \( v_R \) and \( \lambda \). Therefore, any coupling not proportional to the Yukawas will involve a charged mediator. Thus, our strategy to reproduce the structure in Eq. (4) will therefore be to require simply
\[ \frac{v_{B-L}}{M_R} \ll \frac{v_{B-L}}{M_S} \Rightarrow \frac{M_S}{M_R} \ll 1. \]  
(6)

Taking together Eqs. (4) and (5), this simply means that the scale of \((B-L)\) breaking is much below the scale of flavour breaking and we still have a \((E_6\) or \(SO(10)\)) singlet field with a mass around that scale. For instance, we can have the flavour symmetry broken at the Planck or GUT scale, while the scale of \((B-L)\) breaking and \( M_S \) are both around the scale of the right-handed neutrinos, several orders of magnitude below. As \( S \) is a singlet under the gauge group, there is no problem to have its mass at this scale. In this way, the “direct” contributions to \( M_R \) (that do not involve a two step seesaw through \( S \)) are sufficiently small and we get very approximately proportionality of \( M_R \) to \( Y^T_L \) on the left and to \( Y_L \) on the right, as required.

To demonstrate the applicability of this mechanism we present a full realization in a realistic \( SU(3) \) flavour model \( \theta \). The basic features of this symmetry are the following. All left-handed fermions \( (\psi_i, \psi^T_i) \) are triplets under \( SU(3)_R \). To allow for the spontaneous symmetry breaking of \( SU(3)_L \) it is necessary to add several new scalar fields, which are either triplets \((\theta_3, \theta_{23}, \theta_2)\) or antitriplets \((\theta_3, \theta_{23}, \theta_2)\). We assume that \( SU(3)_L \) is broken in two steps. The first step occurs when \( \theta_3 \) and \( \theta_3 \) get a large vev of the order of the mediator scale, \( M_R \), breaking \( SU(3)_L \) to \( SU(2) \). Subsequently a smaller vev of \( \theta_{23} \) and \( \theta_{23} \), of order \( \lambda_2 \), the Cabibbo angle in the down sector and \( \lambda_2^2 \) in the up sector, breaks the remaining symmetry. Moreover in the minimization of the scalar potential the fields \( \theta_{23} \) and \( \theta_{23} \) get equal vevs in the second and third components. Notice that this is precisely the required feature to reproduce maximal atmospheric mixing from Eq. (4). Effective Yukawa couplings are obtained through the Froggatt–Nielsen mechanism \( \theta \). Third-generation Yukawa couplings are generated by \( \theta_{23}, \theta_{23} \) while couplings in the 2–3 block of the Yukawa matrix are always given by \( \theta_{23}, \theta_{23} \). The couplings in the first row and column of the Yukawa matrix are given by \( \epsilon_{ijk} (\theta_{23}, \theta_{23}, \theta_{23}) \). Additional global symmetries are usually imposed to forbid unwanted terms in the effective superpotential, such as a mixed \( \theta_{23}, \theta_{23} \) term, which is forbidden by a discrete \( Z_2 \), for instance.

In this framework, we add a new \( SU(3) \) triplet \( \lambda \) with \( L = -1 \) that gets a vev breaking lepton number. Therefore, a coupling \( (v^T_R) (Y^T_L) (\lambda^i) \) \( S \) is automatically allowed by the symmetries. We must now ensure that \( (\lambda)^T = (1/\sqrt{2}, -1/2, 1/2) \) to reproduce the phenomenologically observed mixings. We do not want to present a full minimization of the scalar potential of the model here, but in fact to obtain this vev is very simple in this model. First we need another field \( \lambda \), antitriplet of \( SU(3) \) to ensure D flatness \( \lambda \). The required vev is obtained if the following \( F \) terms are present in the scalar potential:
\[ F_1 = (\theta_{23}, \lambda), F_2 = (\lambda, \theta_{23}) \quad \text{and} \quad F_3 = (\lambda, \lambda) \]  
(7)

Then, this contribution would give rise to an effective coupling \( (v^T_R) (Y^T_L) (\lambda^i) (\lambda^j) (\lambda^l) (Y_L)^j (v^T_R) \). This means that the third term in Eq. (4) is given exactly by \( (\lambda) \) \( (\lambda)^T \). We still need the other two contributions in Eq. (4) to make the full matrix invertible. In our model, we have other \( SU(3) \) triplets that may also couple to \( (v^T_R) (Y^T_L) \). For instance requiring lepton number conservation, we could also have \( (v^T_R) (Y_L)^j (\theta_{23}, \lambda) \). This gives rise to the first matrix in Eq. (4) as \( \langle (\theta_{23}) \rangle \langle (\theta_{23}) \rangle \) \( T \) with a coefficient of, at least, \( \langle \theta_{23}, \lambda \rangle^2 \).

The easiest way to generate the second matrix would be to introduce a new flavon field \( \lambda' \) carrying lepton number and having a vev orthogonal to \( \lambda \) and \( \theta_{23} \). Still, it is possible to do it only with the fields already present in our \( SU(3) \) flavour theory without any other \((B-L)\)-violating vev. The trick to do this is to notice that rows and columns of the matrices in Eq. (4) are orthogonal if they do not belong to the same matrix. So, if we want to preserve maximal atmospheric mixing and we do not mind perturbing slightly maximal solar mixing, it is sufficient to use another vector orthogonal to \( \theta_{23} \), even though it is not orthogonal to \( \lambda \). Only with the fields we have in the model it is not possible to construct a vev simultaneously orthogonal to \( \theta_{23} \) and \( \lambda \), which would give exact bimaximal mixing. However, the combination \( (\theta_{23}, \lambda) \epsilon_{ijk} \theta_{23}^k \theta_{23}^j \) is a vector in the direction \((1,0,0)\). Now using an interference, \( \langle \lambda_i \rangle \epsilon_{ijk} \theta_{23}^k \theta_{23}^j \langle \lambda_l \rangle \langle \lambda_j \rangle \langle \lambda_k \rangle \).
with a coefficient \((\theta_3, \lambda)\), we obtain the matrix:

\[
\begin{pmatrix}
\frac{\sqrt{2}}{2} & -\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 0 & 0
\end{pmatrix}.
\]

(7)

This structure is very similar to the difference between the second and third matrices in Eq. (6), the only variation is the (1,1) element, and all its rows and columns remain perfectly orthogonal to the first matrix in Eq. (4).

Defining \(\tilde{m}_2, \tilde{m}_1, \tilde{m}_3\) as the mass scales associated to the matrix in Eq. (7) and the first and third matrices in Eq. (4) respectively, and using \(x \equiv \tilde{m}_1/\tilde{m}_2\) and \(y \equiv \tilde{m}_1/\tilde{m}_3\) we have:

\[
\chi_\nu^{-1} = \frac{1}{\tilde{m}_1} \begin{pmatrix}
\frac{1}{2} + \sqrt{2}x & -\frac{1}{2} & \frac{\sqrt{2}}{2} + \frac{1}{2}
\\
-\frac{1}{2} - \frac{\sqrt{2}}{2} & \frac{1}{2} + \frac{\sqrt{2}}{2} & \frac{1}{2}
\\
\frac{1}{2} - \frac{\sqrt{2}}{2} - \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}.
\]

(8)

Now, we can use the fact that the rotation in the (2,3) sector is still maximal and undo this rotation:

\[
R_{23} \left(\frac{\pi}{4}\right) \cdot \chi_\nu^{-1} \cdot R_{23}^\dagger \left(\frac{\pi}{4}\right) = \frac{1}{\tilde{m}_1} \begin{pmatrix}
\frac{1}{2} + \sqrt{2}x & -\frac{1}{2} & \frac{\sqrt{2}}{2} + \frac{1}{2}
\\
-\frac{1}{2} - \frac{\sqrt{2}}{2} & \frac{1}{2} + \frac{\sqrt{2}}{2} & \frac{1}{2}
\\
\frac{1}{2} - \frac{\sqrt{2}}{2} - \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}.
\]

From there, it is straightforward to diagonalize the (1,2) submatrix and we obtain, to order \(x^2\), two eigenstates of masses \(m_1^{-1} = \tilde{m}_1^{-1} \left(1 + \sqrt{2} x + \frac{x^2}{2}\right)\) and \(m_2^{-1} = \tilde{m}_1^{-1} \frac{x^2}{2}\) and \(\sin \theta_{12} = \frac{1 + \sqrt{2} x + \frac{x^2}{2}}{\sqrt{2}}\). Therefore \(x\) must be small to preserve large solar mixing angle. The solar and atmospheric mass differences \(\Delta m^2_{\odot}\) and \(\Delta m^2_{\text{atm}}\) are trivially obtained from the absolute scale and, using \(m_3 = \tilde{m}_3\), from the relation \(m_2/m_3 = 2y/x^2 \approx 1/6\), which fixes \(y = x^2/12\). Now, taking also into account the charged lepton rotation, the difference from maximal solar mixing fixes the value of \(x\); therefore, we give a prediction for the mass of the lightest neutrino. Furthermore, given the experimental values for neutrino mass differences and angles, in this example we can only obtain normal neutrino hierarchy. Moreover before the rotation from the charged lepton sector is taken into account the \(\theta_{13}\) angle is naturally zero, if we want to maintain maximal atmospheric mixing. In this model, we expect a non-negligible \(\theta_{12}^\prime\) rotation in the charged lepton sector, which would give rise to a \(\theta_{13} = \theta_{13}^\prime \sqrt{2}\) (in simple models with a Georgi-Jarlskog factor \(\lambda\) we would expect \(\theta_{12}^\prime = \lambda \theta_3/3\), but other relations are possible \([10]\)). Similarly right-handed neutrino masses can also be fixed, although they depend on the Dirac neutrino Yukawas, which we did not have to specify so far. Full phenomenological details will be presented elsewhere. Finally we have to ensure that these contributions are really the dominant flavon contributions to neutrino masses through this mechanism. In particular, a contribution proportional to \(\overline{\theta}_3\) would spoil our predictions if it were not sufficiently suppressed but this can easily be done with the help of a discrete symmetry forbidding this term.

We have constructed a new mechanism to explain naturally neutrino masses and mixing angles consistently with charged fermion Yukawas. This mechanism is particularly simple in the framework of an SU(3) flavour theory. In this model we have been able to obtain maximal atmospheric mixing, large although not maximal solar mixing, and correct mass differences. As a bonus, we predict that the angle \(\theta_{13}\) comes entirely from the charged lepton sector and the lightest neutrino mass is fixed in terms of the known parameters.


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* Dedicated to the memory of my father, M. Vives


[10] S. Antusch, S. F. King and R. N. Mohapatra,