RIGOROUS THEORETICAL CONSIDERATIONS ON HIGH ENERGY SCATTERING

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1. INTRODUCTION

This review talk will not attempt to encompass all the works which rigorously derive certain results from well-defined hypotheses. It will be restricted to some consequences of the so-called axiomatic field theory, except for the case of lower bounds on scattering amplitudes, in which much stronger assumptions are necessary.

I shall, for definiteness, start with a local field theory in the sense of Wightman \(^1\), \(^2\) (although some of the results have more general validity) and assume moreover:

1) A unique vacuum \(\Omega\) and a minimum mass \(> 0\) for all states orthogonal to \(\Omega\).

2) Each stable particle has an isolated mass hyperboloid (at least in the sector corresponding to its quantum numbers) and the restriction of the representation of the Poincaré group to the corresponding subspace is irreducible. Each stable particle is represented by at least one local field (i.e., at least one local field has non-vanishing matrix elements between the one-particle states and \(\Omega\)).

3) The asymptotic states are complete.

4) The "generalized retarded functions" (g.r.f.) can be defined as tempered distributions with "sharp" support properties.

Assumption 4) is unnecessary for most of the results mentioned below. It may, hopefully, turn out to be a theorem - see \(^3\).
Then the \( S \) matrix is unitary and its matrix elements can be represented by reduction formulae (some precautions are necessary when five or more particles are involved). These complications do not arise for the four-point function, as shown by Hepp \(^4\). They never arise in the theory of Araki and Haag \(^5\). For the process \( 1+2 \rightarrow 3+4 \)

\[
< \rho_3, \rho_4 | S - 1 | - \rho_1, - \rho_2 > = \left\{ \frac{T}{\gamma_j (\gamma_j^2 - m_j^2)} \right\} \gamma_4 (\rho_1, \ldots, \rho_4),
\]

\[
S \left( \sum_{i=1}^{\gamma_4} \rho_i \right) \gamma_4 (\rho) = \frac{1}{(2\pi)^\kappa} \int e^{i \frac{\gamma_4}{\epsilon_j} \cdot x_j} \gamma_4 (x_4, x_5, x_6, x_7) \, dx_4 \ldots \, dx_7,
\]

\[
\gamma_4 (x) = \sum_{\rho} \theta (x_{4} - x_{p_1}) \theta (x_{p_1} - x_{p_2}) \theta (x_{p_2} - x_{p_3}) \times
\]

\[
\times (\Omega, [ A_4 (x_4), A_{p_1} (x_{p_1}), A_{p_2} (x_{p_2}), A_{p_3} (x_{p_3}) ] \Omega)
\]

Other such functions can be used instead of \( r_4 \). The complete list of all generalized retarded functions (g.r.f.) has been worked out in 1959 by several authors (for a short review, references and recent progress, see \(^3\)). In \( p \) space they are the boundary values of a single analytic function denoted

\[
\left\{ \frac{T}{\gamma_j (\gamma_j^2 - m_j^2)} \right\} H (\rho)
\]

**Primitive domain of analyticity: linear program**

The function \( H \) is at first known to be analytic in a domain \( D \) (primitive domain) composed of several conical tubes and of some open sets through which these tubes communicate. \( D \) is star-shaped with respect to \( 0 \) and contains \( 0 \). The linear program consists in finding (as much as one can) the envelope of holomorphy \( \mathcal{H} (D) \) of \( D \), account being taken of the Steinmann identities, which are linear relations between certain g.r.f. It can be shown that :
a) $\mathcal{H}(\mathcal{D})$ is one-sheeted
b) $\mathcal{H}(\mathcal{D})$ is invariant under the complex homogeneous Lorentz group, even if one does not assume any covariance properties of the g.r.f.; it is a purely geometric property of $\mathcal{D}$.

$\mathcal{H}(\mathcal{D})$ is the same for the theories of Araki-Haag and Jaffe. In the case of a Wightman theory it follows from assumption 4) that $\mathcal{H}(\mathcal{D})$ is polynomially bounded in $\mathcal{D}$. In all completions achieved so far, this polynomial boundedness is preserved and it is very likely that this is true in the whole of $\mathcal{H}(\mathcal{D})$.

2. RESULTS OBTAINED IN THE LINEAR PROGRAM FOR THE FOUR-POINT FUNCTION

Notations: $s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$, $u = (p_1 + p_4)^2$

1 - Fixed dispersion relations

$\mathcal{H}(\mathcal{D})$ contains the points of the mass-shell for which $t$ is real, $t_0 \leq t \leq 0$ and $s$ varies in the complex plane minus the $s$ and $u$ cuts, provided the various masses of the theory satisfy certain inequalities. These hold for

$\pi \pi \rightarrow \pi \pi$; $\pi \pi \rightarrow \pi \pi$; $\pi \pi \rightarrow \pi \pi$; $\pi \pi \rightarrow \pi \pi$;

$\kappa \kappa \rightarrow \kappa \kappa$; $\sigma \wedge (\Sigma) \rightarrow \pi \wedge (\Sigma)$.

In a Wightman theory with assumption 4), finitely subtracted dispersion relations hold for these cases. Refs. 4), 7)-11).

2 - Lehmann ellipses (elastic case)

The boundary of $\mathcal{H}(\mathcal{D})$ contains the points of the mass shell for which $s$ is real $> (m_1 + m_2)^2$ and $\cos \theta_s$ is in the ellipse with foci $\pm 1$ and right extremity
\[ \cos \theta_o = \left[ 1 + \frac{(m_1^2 - m_2^2)(M_3^2 - M_4^2)}{k^2 (A - (m_1 - M_4)^2)} \right]^{\frac{1}{2}} \]

(At these points the amplitude is a tempered distribution in \( s \) and an analytic function in the other variables.)

The "absorptive part" \( A_s(s,t) \) is analytic in \( t \), for \( s \) real positive, when \( \cos \theta_s \) is in the large Lehmann ellipse, with foci \( \pm 1 \) and extremity

\[ \cos 2\theta_o = 1 + \frac{2(M_3^2 - M_4^2)(M_3^2 - M_4^2)}{k^2 (A - (m_1 - M_4)^2)} \]

Here, \( M_1 \) and \( M_2 \) are two threshold masses for states with the quantum numbers of particles 1 (or 3) and 2 (or 4), respectively.

\[ \cos \theta_s = 1 + \frac{t}{2k^2} \quad ; \quad k^2 = \frac{1}{4A} \left[ A - (m_1 + m_3)^2 \right] \left[ A - (m_1 - m_2)^2 \right] \]

3 - Mandelstam and Lehmann domains

Mandelstam \(^{12}\) showed that the preceding results can be used to prove that the \( \pi \pi \) scattering amplitude (for example) is analytic (on the mass shell) in both \( s \) and \( t \) in the domain:

\[ |t| < 256 \mu^4 \]

\[^{12}\mu = \text{pion mass}\]

\[^{13}\text{Lehmann also obtained a domain of analyticity (on the mass shell) in} \ s \ \text{and} \ t, \ \text{which interpolates between the dispersion relations and the Lehmann ellipses} \]
4 - Quasi-cut plane

In all cases of masses, $\mathcal{K}(D)$ contains the points of the mass-shell for which: $t$ is real $< 0$, $s \notin s$ cut, $u \notin u$ cut, and $|s| > R(t)$ (i.e., a cut-plane in $s$ minus a finite, although uncomfortably large potato); for the equal-mass case, $R(t) \sim C/\varepsilon^2 |t|^{3+\varepsilon}$,

$$C = \left[ \frac{4}{\pi^4} (M^2 - m^2)^2 \right]^{1+\frac{\varepsilon}{2}}$$

for $t \to -\infty$. ($m = \text{particle mass}, M = \text{threshold}$).

5 - Cut neighbourhoods of physical points

Each physical real point has a complex neighbourhood (in all variables) which is contained in $\mathcal{K}(D)$ except for these points lying on the relevant ($s$ or $t$ or $u$) cut. This is also true at thresholds.

6 - Enlargement of Lehmann ellipse at threshold; Bessis and Glaser's result

This is the first recent result to be mentioned. The preceding point suggests that the shrinking of the Lehmann ellipse at threshold can be remedied and a simple calculation demonstrates that it is a kinematical freak. Bessis and Glaser, by using more variables than Lehmann, have proved, for example in the case of $\pi N$ or $\pi \Lambda$ scattering, analyticity in $t$ in a neighbourhood of the real segment $-3\mu^2 \leq t \leq 4\mu^2$ when $s$ is real and

$$M^2 + \sqrt{2M^2 - \mu^2} < A < (M + \mu)^2$$

($M = \text{baryon mass}, \mu = \text{pion mass}$).

7 - Validity of these results; growth properties

All these results are still true in Araki and Haag's theory of local observables, in Jaffe's theory of "non-renormalizable" local fields. In the latter theory (and probably also in the former) the linear program gives in the domains described

$$|F(\xi, \varepsilon)| < c(\varepsilon) \varepsilon^{\varepsilon(1/4 + 1/\varepsilon)}$$
on the mass shell, for any $\epsilon > 0$. In Wightman's theory with "non-sharp" g.r.f. one gets polynomial boundedness on the mass-shell, off the mass-shell with "sharp" g.r.f. (unitarity improves these results; see below). No properties of covariance of $H(p)$ need to be used in proving the preceding results, so that they hold for arbitrary spin.

8 - First applications

The analyticity provided by 4 and growth properties mentioned in 7 can be used to prove the Pomeranchuk theorems, provided some additional assumptions are made. Many versions have been given by various authors, some rather sophisticated. I quote only one as an example.¹⁷ Let

$$\lim_{s \to \infty} \frac{F(s, t, o)}{s \log s} = 0;$$

if

$$\lim_{s \to \infty} \left( \sigma_{tot}(A, B) - \sigma_{tot}(A, B) \right)$$

exists (perhaps $\infty$) then it is equal to 0. For other versions, see e.g.¹８.

Moreover Khuri and Kinoshita¹⁹ (see also Meiman²⁰) have derived (from forward dispersion relations) certain inequalities which are, in principle, open to experimental verification.

3. USE OF POSITIVITY AND UNITARITY: THE WORK OF MARTIN

(The discussion will be restricted to elastic processes with spinless particles until further notice.) The $S$ matrix being a unitary operator, we can write for

$$A \geq (m_1 - m_2)^2 \quad \text{and} \quad -1 \leq \cos \theta \leq 1,$$

$$F(s, t) \equiv f(s, \cos \theta) = \frac{\sqrt{s}}{k} \sum_{\ell=0}^{\infty} \frac{(2\ell+1) P_{\ell} (\cos \theta) f(s)}{2\ell+1}$$

(1)
\[ A_s (A, t) \equiv a_s (A, \cos \theta) = \frac{1}{k} \sum_{n=0}^{\infty} (2n+1) P^n \cos \theta \Re \frac{1}{z^2} (A) \quad (2) \]

where \( f_\ell (s) \) is a bounded measurable \( \mathbb{L}^2 \) function and

\[ |f_\ell (A)|^2 \leq \Re f_\ell (A) \quad (3) \]

and, in the elastic region:

\[ |f_\ell (A)|^2 = \Re f_\ell (A) \quad (4) \]

Since \( F(s,t) \) and \( A_s (s,t) \) are analytic in \( t \) in their Lehmann ellipses, Eqs. (1) and (2) continue to converge there in the sense of distributions in \( s \). (In fact when \( \cos \theta \) is in the small Lehmann ellipse, Eq. (1) converges in the sense of locally \( \mathbb{L}^2 \) functions of \( s \).)

From the positivity of the metric alone it is easy to derive the following positivity conditions:

For \(-1 \leq \cos \theta \leq 1 \) and physical \( s \):

\[ \left| \frac{d^n}{(d \cos \theta)^n} a_s (A, \cos \theta) \right| \leq \left| \frac{d^n}{(d \cos \theta)^n} a_s (A, \cos \theta) \right|_{\theta = 0} \quad (5) \]

and similar inequalities in the \( u \) channel. \([5]\) also holds off the mass shell.

Martin\(^{21}\) has used these inequalities to enlarge the mass-shell analyticity domain given by the linear program. His method and results are so important that I think worth while to indicate very sketchily the principle of his approach. Neglecting the \( u \) cut and possible subtractions, let us assume that, for \( t_0 \leq t \leq 0 \),
\[ F(A, t) = \frac{i}{\pi} \int_{S} A_{s}(A', t) \frac{ds'}{A' - A} \quad , \quad S = (m_{0}m_{i})^{2} \quad (6) \]

Then, for real \( s \leq S \)

\[ \frac{d}{dt} F(A, t) \bigg|_{t=0} \geq \lim_{\epsilon \to 0} \lim_{\tau \to 0} \frac{i}{\pi} \int_{S} \frac{A_{s}(A', 0) - A_{s}(A', -\tau)}{\tau (A' - A)} ds' - \epsilon \]

For any \( \epsilon > 0 \), there is a \( \tau(\epsilon) > 0 \) such that, for any \( \tau < \tau(\epsilon) \),

\[ \frac{d}{dt} F(A, t) \bigg|_{t=0} \geq \lim_{\epsilon \to \infty} \frac{i}{\pi} \int_{S} \frac{A_{s}(A', 0) - A_{s}(A', -\tau)}{\tau (A' - A)} ds' - \epsilon \]

if \( \tau \) is taken sufficiently small it corresponds to a physical value of \( t \) for all \( s > b \) so that, by (5),

\[ \frac{d}{dt} F(A, t) \bigg|_{t=0} \geq \lim_{\epsilon \to \infty} \frac{i}{\pi} \int_{S} \frac{A_{s}(A', 0)}{A' - A} ds' - \epsilon \]

hence

\[ \frac{d}{dt} F(A, t) \bigg|_{t=0} \geq \frac{i}{\pi} \int_{S} \frac{\frac{d}{dt} A_{s}(A', 0)}{A' - A} ds' \]

Since the right-hand side increases with \( b \) it has a limit. It follows that, as \( b \to +\infty \),

\[ \frac{i}{\pi} \int_{S} \frac{\frac{d}{dt} A_{s}(A', t)}{A' - A} ds' \leq \frac{i}{\pi} \int_{S} \frac{\frac{d}{dt} A_{s}(A', t)}{A' - A} ds' \]

tends to zero uniformly for \( t_{0} \leq t \leq 0 \), which justifies differentiating (6) under the integral sign. Thus,

\[ \frac{d^{n}}{dt^{n}} F(A, t) = \frac{i}{\pi} \int_{S} \frac{\frac{d^{n}}{dt^{n}} A_{s}(A', t)}{A' - A} ds' \quad (7) \]

Now for any \( s \) outside of the cut,
\[
\frac{1}{\pi} \int_{\mathcal{S}} \frac{ds'}{A' - A'} \left| \frac{d^n}{d\epsilon^n} A_n(A', \epsilon) \right| \leq M(A, A_1) \frac{d^n}{d\epsilon^n} F(A_1, \epsilon)
\]

which shows that (7) holds when \( s_1 \) is replaced by \( s \) and that the Taylor series for \( F(s, t) \) as a function of \( t \) in the neighbourhood of 0 is majorized by that of \( F(s_1, t) \). The latter, which has positive coefficients, is known to converge in some disk \( |t| < R \); hence \( F(s, t) \) is analytic in \( |t| < R \) (provided \( s \) is not on the cut) and

\[
F(A, \epsilon) < M(A, A_1) \frac{d^n}{d\epsilon^n} F(A_1, \epsilon) \quad (|\epsilon| < R)
\]

The preceding arguments must be understood in the sense of distributions in \( s \), or carried out after regularizing in \( s \). This presents no difficulty.

The complications introduced by the left-hand cut and subtractions are by no means trivial. They have been overcome by Martin who proved:

Theorem

If the scattering amplitude \( F(s, t) \) obeys a finitely subtracted dispersion relation, and if, for some real \( s_1 \) outside the cuts, \( F(s_1, t) \) is analytic for \( |t| < R \), then \( F(s, t) \) is analytic in the domain

\[
\left\{ s, t : |t| < R, \; s \notin s\text{-cut}, \; u \notin u\text{-cut} \right\}
\]

The number of subtractions is conserved in this domain if even, and increases by at most one if odd.

An immediate consequence, due to (5) is that, for real \( s \geq (m_1 + m_2)^2 \), \( A_s(s, t) \) is analytic in \( t \) in the ellipse with foci \(-4k^2\) and 0, and extremity \( R \). A similar property holds for \( A_u(u, t) \). These ellipses have a non-empty intersection which is a domain of validity of dispersion relations.
Determination of $R$: the method of Sommer

Sommer noticed that the method can be applied to the function obtained by subtracting finite lengths of the $s$ or $u$ cuts from $F(s,t)$, and that $s_1$ can then be chosen $\geq (m_1 - m_2)^2$. This has three advantages:

1) $R$ can now be obtained from the Lehmann ellipses (one chooses $s_1$ to have it as large as possible).

2) Unphysical parts of the cuts, when they occur, can be eliminated.

3) The method can be extended to cases when no dispersion relations have been proved and one only has analyticity in a "quasi-cut" plane.

The result is that $R = 4m^2_{\pi^\pi}$ for $\pi\pi$, KK. For $\pi N \to \pi N$, $\pi\Lambda \to \pi\Lambda$, $R = 4m^2_{\pi^\pi}$ but one must use the result of Bessis and Glaser already quoted.

The only disturbing feature of Martin's use of positivity is the essential role played by polynomial boundedness at infinity. On the mass-shell, polynomial boundedness can be proved even in the case of a Jaffe theory (and probably also in a Araki-Haag theory), so that Martin's theory applies. But this ceases to be true off-the-mass-shell, where positivity continues to hold. Glaser has developed an extensive and very attractive theory of positivity which does not need polynomial boundedness, but is difficult to apply in practice. He has proved, among other things:

**Theorem**

Let $A(p,\bar{p})$ be a function defined and $\mathcal{C}^0$ for $p \in D$ (domain in $\mathcal{C}^N$), such that for every finite sequence $\{\alpha_{\lambda}\}$,

$$\sum_{\alpha,\beta} \frac{\partial^{\alpha+\beta}}{\partial p^\alpha \partial \bar{p}^\beta} A(p,\bar{p}) \geq 0$$

in $D$.

Then there is a function $A(p,q)$ defined and holomorphic in $p$ and $q$ for $p \in D$, $q \in \bar{D}$ such that $A(p,\bar{p})$ is the original function.
4. FURTHER USE OF UNITARITY BY MARTIN \(^{(23), (24), (25)}\) \((\chi - \chi\) CASE)

By permuting the domain obtained previously and by analytic completion one gains new analyticity. But Martin proceeds in a more efficient manner. He subtracts as much as he can of the absorptive parts before making analytic completions. For this purpose it is advantageous to have as much analyticity as one can for the absorptive parts. Martin uses in particular 4) and, here, one is obliged to make a new assumption \(^{(26)}\).

The principle one wants to use is this: from the analyticity of \(f(s, \cos \theta)\) in the ellipse with extremity \(\cos \theta_1 = 1 + 8/(s-4)\),

\[
\limsup_{\ell \to \infty} | f_\ell (s) |^{1/\ell} = \left[ \cos \theta \pm \sqrt{\cos^2 \theta - 1} \right]^{-1} \tag{8}
\]

and we are tempted to conclude that, in the elastic region

\[
\limsup_{\ell \to \infty} \left[ \Im f_\ell (s) \right]^{1/\ell} = \limsup_{\ell \to \infty} | f_\ell (s) |^{2/\ell} = \left[ \cos \theta \pm \sqrt{\cos^2 \theta - 1} \right]^{-1} \tag{9}
\]

which would correspond to \(t = 16 + 64/(s-4)\).

Unfortunately 8) is true only in the sense that

\[
\limsup_{\ell \to \infty} \left[ \int_a^{a+\varepsilon} f_\ell (s') ds' \right]^{1/\ell} = e^{i \frac{\theta}{2}} \quad (\theta \approx \theta_2) \tag{\theta_2 \sim \theta_2}
\]

from which one cannot deduce a similar relation for \(\Im f_\ell\). (The derivation would work in the opposite direction thanks to Schwarz's inequality.) The argument would, of course, be valid if \(f_\ell (s)\) were a continuous function, and one immediately tries to invoke second sheet analyticity. But the latter itself can be proved, as Martin has shown by instructive counter-examples, only at the cost of a specific assumption. It becomes true, together with 9) if one makes the following, intuitively acceptable, but completely ad hoc hypothesis: \(\sigma_{\text{tot}} (s)\) is bounded in any subinterval of \(4 < s < 16\).
I cannot give here a description of Martin's completions, which require an astonishing display of ingenuity. The real sections of the domain obtained are given in Fig. 1 and the partial wave domain appears in Fig. 2.

This domain is not final (it is not natural), but counter examples show that one cannot hope to obtain the Mandelstam representation without more information. These counter-examples have an independent interest since they give rise to interesting integral representations, but I refer the reader to the above references.

Another result of this work is that "quasi-dispersion relations" hold when \( t \) is inside the parabola with focus 0 and extremity 4.

5. PROISSART BOUNDS AND RELATED TOPICS

The bounds established by Froissart in 1961 as a consequence of Mandelstam's representation (with polynomial boundedness) now follow from axiomatic field theory thanks to the results of Martin described in Section 3.

Let \( w(s) \) be a positive \( C^\infty \) function vanishing outside of the interval \( [0, \epsilon] \), with \( \int w(s)ds = 1 \). Denote

\[
\tilde{\alpha}_a (s, \cos \theta) = \frac{\sqrt{4}}{k} \int w(s') \frac{k'}{k'} \alpha_a (s', \cos \theta) ds',
\]

This function is analytic in an ellipse with extremity \( 1 + \frac{R}{2k^2} \), so that

\[
\frac{\sqrt{4}}{k} \sum_{\ell} (2\ell + 1) \mathcal{P}_{\ell} (1 + \frac{R}{2k^2}) \mathcal{w}\mathcal{w}^{\mathcal{r}} \mathcal{I}_m f_{\ell}^r (s) = \tilde{\alpha}_d (s, 1 + \frac{R}{2k^2}) < \text{const.} A^N
\]

\[
(2\ell + 1) \mathcal{w}\mathcal{w}^{\mathcal{r}} \mathcal{I}_m f_{\ell}^r (s) \leq \text{const.} A^N (1 + \frac{s}{A})^{-\ell}
\]
\[ \frac{\sqrt{2}}{k} \sum_{\ell = \ell_0}^{\infty} (2\ell + 1) \omega \ast \text{Im} f_\ell (A) \leq \text{const.} \, A^{N/2} (1 + \frac{\xi}{\sqrt{A}})^{2\ell + 3} e^{-L \log (1 + S)} \]

This remainder can be made as small as one wants, and polynomially decreasing in \( s \), by choosing \( L \geq \lambda \sqrt{s} \log s \) with large enough \( \lambda \). It follows that

\[ A_\ell (A, 0) \sim (\ell + 1) \frac{\sqrt{A}}{k} \leq \text{const.} \, A (\log A)^2 \]

As far as the amplitude is concerned, we have, by Schwarz's inequality,

\[ |w \ast f_\ell (A)|^2 = \left| \int w(s - s') f_\ell (s') ds' \right|^2 \leq \int |f_\ell (A')|^2 w(s - s') ds' \leq w \ast \text{Im} f_\ell (A) \leq \text{const.} \, A^{N/2} (1 + \frac{S}{\sqrt{A}})^{-\ell}(2\ell + 1)^{-1}, \]

\[ |w \ast f_\ell (A)| \leq \text{const.} \, A^{N/2} (1 + \frac{S}{\sqrt{A}})^{-\ell/2}(2\ell + 1)^{-1/2} \]

from which it also follows that the contribution to the expansion 1) of the terms with \( \ell > L' = \lambda' \sqrt{s} \log s \) is negligible for a suitable choice of \( \lambda' \). This, together with \( |f_\ell (s)| \leq 1 \), is the basis of a great number of bounds on the amplitude and its absorptive part.

A similar technique is used to prove that, even in a theory (such as Jaffe's) where polynomial boundedness is replaced by

\[ |F(s, \xi)| \leq C(\xi) \exp (-A s + /\xi) \]

for every \( \xi > 0 \), one has in fact polynomial boundedness and twice subtracted dispersion relations \( ^{25})^{28} \). One starts from bounds which analyticity in the Lehmann ellipse already yields (Greenberg and Low \( ^{29} \)):
\[ \left| \sum_{\ell=L}^{\infty} (2\ell+1) f^{(\ell)} \right| < \text{const. } (\frac{A}{\epsilon} + 1)^2 (2L+3) e^{-L \log \left( 1 + \frac{C}{\epsilon} \right) \sqrt{C(t)}} e^{\frac{\epsilon}{2} L^2} \]

which can be made arbitrarily small for \( L \sim \lambda s^2 \). Hence, for \( t < 0 \), \(|F(s,t)| < C s^4\) along the real axis (in the sense of distributions). From the Phragmén-Lindelöf theorem, polynomial boundedness follows, and the number of subtractions is actually at most two by Froissart bounds.

A short selection of bounds

1 - Froissart bounds

\[ |F(A,0)| < \text{const. } A \left( \log \frac{A}{\lambda} \right)^2 \]

\[ \sigma_{\text{Tot}}(A) = \frac{4\pi}{k\sqrt{s}} A s (A,0) < \frac{\pi}{m^2} \left( \log \frac{A}{\lambda} \right)^2 \quad (R = \sqrt{m} \sqrt{\lambda}) \]

(this recent value of the constant is due to Lukaszuk and Martin, 1967, to appear).

Fixed \( t \) and fixed angle upper bounds are obtained using

\[ |P_{\xi \cos \theta}| < (\xi \sin \theta)^{-\frac{1}{4}} \quad \text{for real } \theta : \]

\[ |F(A,t)| < \frac{\text{const.}}{|t|^{1/4}} A (\log A)^{3/2} \quad (t < 0) \]

\[ |f(A,\cos \theta)| < \text{const.} \frac{A^{3/4} (\log A)^{3/2}}{\sqrt{\sin \theta}} \]

Kinoshita, Loeffel and Martin (K.L.M.) have shown that if Mandelstam's representation holds, the last bound can be much improved:

\[ |f(A, \cos \theta)| < \text{const.} \frac{(\log A)^{3/2}}{\sin^2 \theta} \]
2 - Miscellaneous

$$\sigma_{\text{el},\text{tr},w} > \text{const.} \frac{\sigma_{\text{tor}}^2}{(\log A)^2} \quad (\text{Martin}^{31})$$

$$\frac{d}{dt} \log A(A, t) \bigg|_{t=0} \geq \left\{ \begin{array}{l}
\frac{1}{8} \left( \frac{\sigma_{\text{tor}}}{4\pi} - \frac{1}{k} \right) \\
\frac{1}{3} \frac{\sigma_{\text{tor}}}{\sigma_{\text{el}}} \left( \frac{\sigma_{\text{tor}}}{4\pi} - \frac{1}{k} \right) 
\end{array} \right. \quad (\text{Martin and Mac Dowell}^{33})$$

$$\Delta t^{-1} = \frac{d}{dt} \log |F(A, t)|^2 |_{t=0} \leq C \left( \log \frac{A}{A_0} \right)^2$$
(This was proved by different methods by Bessis$^{34}$ and Kinoshita$^{35}$ assuming that

$$\left| \frac{\text{Im} F(A, t)}{\text{Re} F(A, t)} \right| > \eta > 0$$

for $-\varepsilon \leq t \leq 0$.)

3 -

In a paper submitted to this Conference by N. Van Hieu, and describing joint work with Logunov, Mestirirshvili and N. ngoc Thuan (L.M.N.N.) (see also 36), the following bounds are proved by using the fact that expansions (1) and (2) essentially stop at $\ell = \lambda \sqrt{\log A}$ and Schwarz's inequality:

$$\frac{d\sigma_{\text{el}}}{d\cos \Theta} \bigg|_{\theta=0} \leq \text{const.} A (\log A)^2 \sigma_{\text{el}} \quad (\text{10})$$

$$\frac{d\sigma_{\text{el}}}{d\cos \Theta} \bigg|_{\theta \neq 0, \pi} \leq \text{const.} \frac{\sqrt{A} \log A}{\sin \Theta} \sigma_{\text{el}} \quad (\text{11})$$

$$\frac{d\sigma_{\text{el}}}{dt} \bigg|_{t=0} \leq \text{const.} (\log A)^2 \sigma_{\text{el}} \quad (\text{12})$$

$$\frac{d\sigma_{\text{el}}}{dt} \bigg|_{t \neq 0} \leq \text{const.} \frac{\log A}{\sqrt{1|t|}} \sigma_{\text{el}} \quad (\text{13})$$
6. THE CERULUS-MARTIN LOWER BOUND AND RELATED TOPICS

1 - Jin-Martin lower bound for forward amplitudes

From forward dispersion relations and positivity (using the theory of Herglotz functions), Jin and Martin proved that

\[ |F_1(\lambda, \sigma)| > \frac{\text{const.}}{\lambda^2} \]

in every non-real direction (the constant depends on the direction). For real \( s \) this is only true in the sense of averages \( [\text{e.g., for } 1/s \int_0^s F(s', t) ds'] \).

2 - Non-forward fixed angle lower bound of Cerulus and Martin (C.M.)

Assume that, for \( s > (m_1 + m_2)^2 \), \( f(s, \cos \theta) \) is analytic in \( \cos \theta \) in a bounded domain \( D(s) \) containing the real interval \( J - \rho, \rho \) (\( \rho = 1 + (m_1^2/2k^2) \)). Let \( u(z) \) be the function, harmonic in \( D(s) \) except on a cut along \( [a, b] \), (with \( 0 < a < b \)), equal to 0 on the boundary \( \partial D \) of \( D(s) \) and to 1 on the cut \( [a, b] \). Then

\[ \log |f_1(\lambda, z)| < a(1) \max_{z \in [a, b]} |f_1(\lambda, z)| + (1 - a(1)) \max_{z \in D} |f_1(\lambda, z)| \]
so that, assuming that \( |f(s,z)| < A s^N \) on the boundary of (and in) \( D(s) \), and taking into account the Jin-Martin lower bound,

\[
M_{\max} \quad |f(s, \cos \theta)| > B e^{-\frac{C \cos \theta}{\cos(z)}} \quad \text{for} \quad |\cos \theta| < \alpha
\]

To obtain the C.M. bound, one must leave the hopefully safe, but not plentiful realm of axiomatic field theory, and assume that \( D(s) \) is quite large, although not as large as the cut-plane of Mandelstam's representation, and that, as \( s \to \infty \), it keeps a fixed shape as shown below:

One then finds \( u(1) \sim C'k \) and

\[
M_{\max} \quad |f(s, \cos \theta)| > e^{-C' \sqrt{\log s}} \quad \text{for} \quad |\cos \theta| < \alpha
\]

Since this bound seems to be making contact with experiment, it is worth while to discuss it a little more.

The question of the domain

Finding \( u(1) \) for various shapes of domains \( D(s) \) only requires a compendium of conformal mappings which will map \( D(s) \) minus cut onto an annulus or other elementary object. For a domain bounded by two arcs of circles passing through \( \pm \) and making an angle with the real axis \( \tilde{D}(s) \) given by:

\[ |\arg( +z/ -z)| \]

one finds \( u(1) \sim O(m/k) \); if
stays fixed one gets, instead of the C.M. bound, \( \exp(-Cs^{\pi/2}\sqrt{s} \log s) \).
If one tries to approximate the domain obtained by Martin, by taking
\( \chi \sim \text{const}/\sqrt{s} \) one gets: \( \exp(-Cs^{\chi} \sqrt{s} \log s) \), not very signifi-
cantly different from 0!

Other examples of shapes of \( D(s) \) : Chiu and Tan\(^{42}\)
take \( D(s) \) to be the angle: \( |\arg(-z)| \) and get again
\( \exp(-Cs^{\chi} \sqrt{s} \log s) \). Eden and Tan\(^{43}\) consider a more complicated
domain making an angle with the real axis, and find once more
the above expression.

It is not, however, the angle which really determines
the bound : for example the domains

![Diagram of two domains](image)

both give similar bounds: \( \exp(-Cs^{\pi/2}\sqrt{s} \log s) \) with different \( C \).

I think that the true conclusion of all this is that the
C.M. bound is extremely sensitive to the shape of \( D(s) \) and to its
rate of shrinking as \( s \to \infty \).

Dependence on the angle

1) – Kinoshita has shown\(^{44}\) that in the C.M. bound \( \exp(-C(\theta) \sqrt{s} \log s) \),
\( C(\theta) \) has a dependence of the form \( \text{const.} \sin \theta \) for small \( \theta \).

2) – The work of Tiktopoulos and Treiman (T.T.)\(^{45}\)

Taking \( D(s) \) as above, these authors assume that

a) in \( D \), \( f(s,z) \) is analytic and

\[
|f(\alpha, z)| < C(\alpha) \left| f + \sqrt{\rho^2 - \varepsilon^2} \right|^{M(\alpha)}
\]

b) \( f(s,z) \) has no zero in \( D \).
Then

\[ G(s, z) = M(A) \log \left( 1 + \sqrt{\rho^2 - z^2} \right) + Q(A) - \log f(A, z) \]

is analytic in \( D \) and has a positive real part. After mapping \( D \) onto the unit circle \( G(s, z) \) gives rise to a Herglotz function obeying a Poisson representation. Defining a new variable \( \lambda \) and a new function \( \psi(\lambda) \) closely related to \( \phi(s, z) = -\log|f(s, z)| \), T.T. find explicitly the necessary and sufficient conditions that the values of \( \psi(\lambda) \) at any finite set of points must satisfy. These are, in principle, open to experimental verification. Example: \( D = \) cut plane, \( f(s, z) = f(s, -z) \). Then for any \( 1 > z_2 > z_1 > -1 \)

\[ \frac{\phi(A, z_2)}{\sqrt{\rho^2 - z_2^2}} > \frac{\phi(A, z_1)}{\sqrt{\rho^2 - z_1^2}}, \]

\[ \sqrt{\rho^2 - z_2^2} \phi(A, z_2) \leq \sqrt{\rho^2 - z_1^2} \phi(A, z_1) \]

Something can still be said if there is a finite number \( N(s) \) of zeros bounded when \( s \to \infty \) or even if \( N(s)/\phi(s, z) \to 0 \) as \( s \to \infty \).

The question of polynomial boundedness

Martin \(^{46} \) has shown that the C.M. bound still holds if polynomial boundedness is replaced by

\[ |F(s, t)| < e^{01t^{3-\varepsilon}} s^{02t^{3-\varepsilon}} \]

\((t > 4, \ s \ \text{physical}) \) and similarly for \( |F(s, u)| \).

7. GENERALIZATIONS (TWO-BODY PROCESSES)

1 - Inelastic two-body processes

Sommer \(^{47} \) has extended Martin's results (Sections III and IV) to such processes.

On the other hand, L.M.N.N. (Ref. \(^{36} \) and paper presented at this Conference) have extended their bounds \((10),(11),(12),(13)\) to
these two-body reactions (so that these inequalities remain true after substituting the symbol $\sigma_{\text{inel}}$ to $\sigma_{\text{el}}$ everywhere).

2 - Non-zero spins

One encounters quite non trivial difficulties in extending the results of Sections III, IV, and the various bounds, to cases with non-zero spin. Martin and Sommer 48),25) have extended the results of III and IV to $\pi N$. The general case has been treated by Mahoux and Martin, and is the subject of a paper appearing in these proceedings, to which the reader is referred.

8. INELASTIC PROCESSES

Finally, I shall say a few words about inelastic processes since the paper of Logunov, Westvirishvili, Nguyen van Hieu and Nguyen ngoc Thuan (L.M.N.N.) submitted to this Conference is largely devoted to them and also because of an interesting paper recently written on this subject by Tiktopoulos and Treiman (T.T.II) 49).

1 - Analyticity properties from the linear program

Ascoli and Minguzzi 50) have generalized Lehmann's derivation of his small ellipse to production amplitudes (see also 36) and proved the existence of an ellipse of analyticity in one variable. (A great difference with the four-point function is that in that case any point of the Lehmann ellipse had a cut neighbourhood of analyticity in all variables. This is by no means the case for the five-point function; only preliminary information is available at present, which will not be discussed here.)

2 - Use of Martin's results and unitarity

T.T. and L.M.N.N. start essentially in the same way. I adopt T.T.'s notations. Consider

$$a + b \rightarrow c + n \quad \text{other particles}$$
In the C.M. frame, let $\theta$ and $\varphi$ be the polar and azimuthal angles of $\vec{p}_c$, the polar axis being $\vec{p}_a$. The kinematical variables being $s, \theta, \varphi$ and $v$ (collective name $l$), denote the phase space element by

$$d\varphi d\cos\theta \rho(s,v) dv$$

and $k = |\vec{p}_a|$. The amplitude as a function of these variables can be written

$$T(s, \theta, \varphi, v) = \sum_{l,m} (2l+1) d_l^m(\theta) \frac{e^{im\varphi}}{\sqrt{2\pi}} T_l^m(s, v)$$

and

$$\frac{d\sigma_{in}}{d\cos\theta} = \frac{i}{k} \int dv \rho(s, v) \sum_{l', m'} (2l'+1)(2l+1) d_{l'}^{m'} \frac{1}{d_l^m} T_l^m T_{l'}^{m'}$$

The contribution of this process to the absorptive part of the elastic process is

$$\int dv \rho(s, v) \sum_{m} |T_l^m(s, v)|^2 \leq \frac{i}{k} \frac{\text{Im} T_l^l(s)}{f_l} (s)$$

($f_l =$ partial wave for $a+b \rightarrow a+b$).

From this, using Schwarz's inequality, one deduces

$$\frac{d\sigma_{in}}{d\cos\theta} \leq \frac{i}{k^2} \left[ \sum_{l} \frac{(2l+1)}{\text{Im} T_l^l(s)} \frac{1}{2} M_l(\theta) \right]^2$$

where

$$M_l(\theta) = \sum_{m} \left| d_l^m(\theta) \right|^2, \quad \text{T.T. find } M_l(\theta) \sim \begin{cases} 1 : \cos\theta = \pm 1 \\
\theta^{-\frac{3}{2}} (\cos\theta \sin^2\theta)^{-\frac{1}{2}} \end{cases}$$

From then on, one can apply exactly the same tricks as for the derivation of bounds on the elastic amplitude. One gets

$$\frac{d\sigma_{in}}{d\cos\theta} \leq C_1 A (\log A)^4 (\cos \theta = \pm 1)$$

$$\text{T.T.}$$

$$\frac{d\sigma_{in}}{d\cos\theta} \leq C_2 A^{2/3} (\log A) \frac{10}{3} (\cos \theta \sin \theta)^{-\frac{1}{3}} (\cos \theta \pm 1)$$
L.M.N.N. derive by similar means
\[
\frac{d\sigma_{\text{in}}}{d\cos \theta} \bigg|_{\theta = 0} \leq \text{const. } A \left( \log s \right)^2 \sigma_{\text{in}}
\]
\[
\frac{d\sigma_{\text{in}}}{d\cos \theta} \bigg|_{\theta \neq 0, \pi} \leq \text{const. } \frac{\sqrt{s} \log s}{\sin \theta} \sigma_{\text{in}}
\]
which also hold for cross-sections summed over several processes producing \( c \).

If one assumes enough analyticity in \( \cos \theta \) for \( \text{Im } f(s, \cos \theta) \)_{\text{inelastic}} (\text{= contribution of the process to absorptive part of elastic process}), L.M.N.N. show that a Cerulus-Martin bound follows (identical technique). On the other hand, T.T., assuming cut plane analyticity in \( \cos \theta \), extend the work of Kinoshita, Loeffel and Martin and find:
\[
\frac{d\sigma_{\text{in}}}{d\cos \theta} \leq \mathcal{C} \sigma_{\text{in}}(s) \left( \frac{\log s}{\sin^2 \theta} \right)^{3/2} \quad (\cos \theta \neq \pm 1)
\]
which is a great improvement since \( \sigma_{\text{in}}(s) \leq \mathcal{C} (\log s)^2 \).

Final apologies

Many interesting topics could not be included here, and I can only apologize and recommend to the interested reader to consult (among others) the following general references, which, in turn, refer to an abundant literature:


3) R.J. Eden - "High energy collisions of elementary particles", Cambridge University Press, 1967. This book contains a large amount of information concerning the subject of this talk as well as all aspects of high energy scattering.
REFERENCES


24) A. Martin - Communication to the Rochester Conference (1967).


26) A. Martin - "Inability of Field Theory to Exploit the Full Unitarity Condition", Preprint CERN Th. 727 (1966).


39) A. Bonnier and R. Vinh Mau - Phys. Letters \textbf{24B}, 477 (1967), and to be published.


FIG. 1  real-$s$-sections of Martin's domain

FIG. 2  Martin's domain for partial waves (s-plane)