DUAL MODELS AND DUAL PHENOMENOLOGY

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1. INTRODUCTION: THE DUALITY FRAMEWORK

The accompanying lectures by A. Donnachie describe how, according to present ideas about strong interaction physics, hadron reactions at high energies \( p_{\text{lab}} \geq 5 \text{ GeV/c} \) are simply described in terms of Regge pole exchange in the t-channel and at low energies \( p_{\text{lab}} \leq 2 \text{ GeV/c} \) in terms of resonance formation in the s-channel. In addition, both the positions of the resonances and of the Regge poles are determined by the same function, the Regge trajectory, so that there exists a correlation between these two dynamical mechanisms. The purpose of these lectures is to investigate this correlation further and examine the consequences of assuming that Regge poles and resonances are just alternative ways of expressing the same dynamical information. The theoretical framework which has been developed around this assumption has come to be known as the duality framework\(^1\). In particular, we shall focus our attention on dual models\(^2-4\), which provide us with explicit analytic functions satisfying duality and which can even be used for phenomenological purposes.

In order to obtain a concrete view of the situation, consider Figs. 1 and 2 showing the measured \( \pi^+p \) and \( \pi^-p \) total cross-sections (proportional to the imaginary part of the forward scattering amplitude by the optical theorem). At low energies \( p_{\text{lab}} \leq 3 \text{ GeV/c} \), many bumps corresponding to resonances are seen; at higher energies \( \sigma_{\text{tot}} \) decreases smoothly with no measurable structure. In addition to the resonances clearly seen in \( \sigma_{\text{tot}} \), there are several resonances which can be identified only by more detailed phase-shift analysis as small circles on the Argand plot (see Donnachie's lectures and Fig. 3). These couple so weakly to the \( \pi N \) system that they do not produce any measurable effect in \( \sigma_{\text{tot}} \).

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Fig. 1 The $\pi^- p$ total cross-section (data from HERA compilation, unpublished).

Each resonance in itself corresponds to a well-defined angular distribution in the decay $N^* \rightarrow \pi N$ (this can be measured, for example, in the elastic formation process $\pi N \rightarrow N^* \rightarrow \pi N$), depending on the spin and parity of the resonance. Thus the differential cross-section of $\pi N \rightarrow \pi N$ at the position of the first $\pi N$ resonance, the $\Lambda(1236)$ with $I^J = \frac{3}{2} \frac{1}{2}^+$, behaves as $d\sigma/d\Omega = 1 + 3 \cos^2 \theta$ in agreement with $J^P = \frac{3}{2}^+$. This is a very pure case; the $\Lambda(1236)$ decays almost 100% to $\pi N$; there is little background and the interference with other resonances is small. When the total c.m. energy is increased to $\sqrt{s} > 1.5$ GeV the density of resonances grows, they overlap and interfere. As a consequence, the differential cross-section for $\pi N \rightarrow \pi N$ is no more determined by a single resonance, but rather the different resonances interfere so as to produce a forward peak, characteristic of Regge pole exchange. In addition, a non-resonant
Fig. 2 The $\pi^+ p$ and $\pi^- p$ total cross-sections with resonance positions indicated.

Fig. 3 The $\pi N$ resonances plotted with mass squared versus spin. The notation is $N$(mass, $J^P$) for $I = \frac{1}{2}$ and $\Delta$(mass, $J^P$) for $I = \frac{3}{2}$. 
background starts to appear in $\pi^+p + \pi^+p$. Neglecting this background (it will soon be correlated with Pomeron exchange at high $s$), we shall assume that all (of the imaginary part) of the amplitude is given by resonances. This is the important assumption of resonance dominance. At still higher energies the different resonances can no longer be separated, and the amplitude is most economically described by the exchange of a few Regge poles (Regge pole dominance). Together with analyticity these two assumptions of resonance and Regge pole dominance form the basis of the duality framework. Both these assumptions are known to be approximate; both very inelastic resonances and low-lying Regge poles are, in practice, irrelevant to them.

Notice that the resonance shape is given by the Breit-Wigner function, which with the abbreviation $\varepsilon = 2(E_R - E)/\Gamma_R$ can be written as

$$\frac{1}{\varepsilon - i} = \frac{\varepsilon}{\varepsilon^2 + 1} + \frac{i}{\varepsilon^2 + 1}.$$ 

Thus the real part vanishes at the resonance position and dominates the resonance tail, while the imaginary part has a maximum at resonance position and a short tail. A resonance contributes, therefore, strongly to the imaginary part of the amplitude and the assumption of resonance dominance can only be extended to the imaginary part.

In order to see qualitatively the consequences of joining the assumptions of resonance and Regge pole dominance, consider, for example, $\pi N$ charge-exchange scattering (here there is no Pomeron exchange). We can expand the imaginary part of either one of the two spin amplitudes as

$$\sum_{N_i, \Delta_i} \frac{\pi^-}{p} \frac{\pi^0}{n} = \sum_i \frac{\pi^-}{p} \frac{\alpha_i}{n} \frac{\pi^0}{n},$$

where the sum at left goes over all resonances and at right over all Regge poles. At low energies, the resonance sum is dominated by a few terms and the Regge pole sum is complicated. For example, at $\sqrt{s} = 1.235$ GeV it
would be meaningless to use the Regge pole expansion. At high energies many overlapping resonances contribute, but the Regge pole expansion is dominated by just one term, the exchange of the $\rho$ Regge pole. At intermediate energies both expansions contain just a few terms and one has approximately

\[
\sum_{\text{strong}} N, \Delta \approx \pi^- \pi^0 \quad \rho \quad \pi^- n^0 \rightarrow p n
\]

In this form it is clear how duality constrains the resonances and Regge poles. The resonances at left have to be so constrained that they just reproduce what is known to follow from the exchange of the $\rho$ Regge pole. For example, they have to reproduce the almost constant shape of $d\sigma/dt$ typical of a Regge pole term.

Historically, both the resonance and the Regge pole descriptions were first treated quite separately and it was only later realized that they could be essentially equal. The original assumption was that the resonance and Regge pole terms are basically different and have to be added in order to obtain the total amplitude (interference model). This is analogous to what is done in perturbation theory. For instance, the lowest order Feynman graphs for electron-positron scattering are

\[
\begin{align*}
\text{and the corresponding amplitudes are added. The interference model has subsequently been generalized and modified and it is clear that some type of interference model can always be constructed to reproduce the predictions of duality for a fixed reaction. However, the over-all picture favours duality, although the issue still is somewhat controversial.}
\end{align*}
\]
The duality between resonances and Regge poles, as described above, cannot be complete since the amplitude may, at low energies, contain considerable background. This is seen clearly in Figs. 4 and 5 showing the measured $K^- p$ and $K^+ p$ total and elastic cross-sections. As far as is known, there are no prominent resonances in the $K^+ p$ system (in the sense of the quark model, $K^+ p$ is an exotic state). However, the imaginary part (Fig. 4) is very definitely non-zero and thus must come from some other source than resonances. A solution to this problem is suggested by the Freund-Harari assumption\textsuperscript{5}), stating that the structureless background in the imaginary part at low energies corresponds to the Pomeron contribution at high energies, similarly as the resonance-dominated imaginary part corresponds to the amplitude given by ordinary Regge poles (i.e., those with $\alpha_0 < 1$). According to all available evidence the Pomeron Regge pole is a very special one: it is the only one remaining at asymptotic energies and its Regge trajectory $\alpha_p(t) = 1 + \alpha' t$ has probably a slope ($\leq 0.4$), which is smaller than that of the normal trajectories ($\approx 0.9$). It is thus natural that it occupies a special position also in the duality framework; it is not an ordinary Regge pole, but rather a way of describing in the language of Regge poles some basically unknown phenomena. In $K^+ p \rightarrow K^+ p$, all of the imaginary part of the amplitude is thus dual to the Pomeron, the imaginary part due to resonances being zero. There is also a real part, due to tails of resonances in the $K^- p$ channel, which accounts for the decrease of the elastic $K^+ p$ cross-section in Fig. 5. The same statements will apply to all reactions involving an exotic direct channel, for example all KN and NN reactions. In particular, if there is a reaction where no Pomeron exchange is possible (e.g., $K^+ n \rightarrow K^0 p$, $K^+ p \rightarrow K^+ \Lambda$, $p p \rightarrow p \Delta$, etc.) the whole amplitude should be real. These predictions are rather hard to test.

In addition to the assumptions of resonance and Regge pole dominance, the third crucial assumption in the duality framework is analyticity, since this establishes a connection between low and high energies or between different channels. Prior to the invention of dual resonance models, analyticity was used rather implicitly (for example, in the form of finite energy sum rules); the dual resonance models, on the other hand, make it possible to carry out the crossing between different channels explicitly. Those models can thus very conveniently be used to illustrate the constraints imposed by duality.
Fig. 4  The $K^+p$ and $K^-p$ total cross-sections.

Fig. 5  The $K^+p$ and $K^-p$ elastic cross-sections.
Crossing refers to going from one channel to another and we shall define these as follows. Consider the two-body reaction $p_a + p_b + p_1 + p_2$, with masses as in $\pi N + \pi N$. The invariant kinematical variables are defined by the equations

$$ s = s_{ab} = (p_a + p_b)^2 $$

$$ t = t_{a1} = (p_a - p_1)^2 $$

$$ u = t_{a2} = (p_a - p_2)^2 $$

and satisfy the constraint $s + t + u = 2m_N^2 + 2m_\pi^2$. There are two independent variables, which can be chosen to be the total centre-of-mass (c.m.) energy $E$ and the c.m. scattering angle $\theta_s$, related to $s$, $t$ and $u$ by

$$ s = E^2 $$

$$ t = -\frac{\lambda(s, m_N^2, m_\pi^2)}{2s} (1 - \cos \theta_s) $$

$$ u = \frac{(m_N^2 - m_\pi^2)^2}{s} - \frac{\lambda(s, m_N^2, m_\pi^2)}{2s} (1 + \cos \theta_s) $$

where $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$. The values of $s$, $t$ and $u$ experimentally accessible are limited by the physical condition $\sin^2 \theta_s \leq 1$; evaluating this from Eq. (1.2), one obtains a boundary of the physical region consisting of two curves

$$ t = 0 $$

$$ su = (m_N^2 - m_\pi^2)^2 $$

and limiting three different regions (cf. Figs. 11 and 12 in Section 2). The region in which $t \geq 4m_N^2$ is the physical region of the $t$-channel reaction $p_a + \bar{p}_1 + \bar{p}_b + p_2$ ($\bar{p} = -p$) or, in the $\pi N$ case, of the reaction $\pi + \pi + p + \bar{p}$. The $u$-channel corresponds to the reaction $p_a + \bar{p}_2 + p_1 + \bar{p}_b$ and is, in the $\pi N$ case, essentially the same as the $s$-channel [notice the symmetry of Eqs. (1.3) under the interchange of $s$ and $u$].

The dynamics of the process $p_a + p_b + p_1 + p_2$, as well as of all the processes connected with this by crossing, are governed in a spinless case with no internal symmetries by the analytic scattering amplitude $A(s, t, u)$. 
If the particles have spin or if some internal symmetry is taken into account, there are more amplitudes associated with different spin and internal symmetry states. Crossing is essentially equivalent to analyticity. This means, by definition in the spinless case with no internal symmetries, the fact that, since $A(s,t,u)$ is analytic, measurable predictions can be computed on the basis of it in any of the three channels. Notice that under crossing $A(s,t,u)$ remains completely unchanged; one just moves to another region of the kinematic $s,t,u$ plane. In addition, if the reaction under consideration exhibits some special symmetries, crossing translates these into symmetry requirements on $A(s,t,u)$ (crossing symmetry). Thus in $\pi^0\pi^0 \rightarrow \pi^0\pi^0$ all the channels are evidently identical so that $A(s,t,u)$ for this reaction has to be invariant under all permutations of $s$, $t$ and $u$. Similarly, in $\pi^+\pi^- \rightarrow \pi^+\pi^-$ (s-channel) the t-channel is also $\pi^+\pi^- \rightarrow \pi^+\pi^-$ but the u-channel is $\pi^+\pi^- \rightarrow \pi^+\pi^+$ so that $A(s,t,u) = A(t,s,u)$. Note the qualitative difference between crossing and crossing symmetry (in the terminology used here); crossing is something very basic (= analyticity) while crossing symmetry is an additional reaction-dependent symmetry.
II. $\pi\pi$ SCATTERING

In order to see what conditions are imposed on the resonances and Regge poles in the duality framework, we shall first consider the theoretically relatively simple case of $\pi\pi$ scattering. According to duality, resonances and ordinary Regge poles are complementary in the description of the imaginary part of the amplitude (we omit the background Pomeron part of the amplitude from consideration in this Section). If there are no non-exotic resonances, the amplitude has to be real and the Regge poles have to be so constrained that the imaginary part of the amplitude vanishes. The $\pi\pi$ system is exotic in the isospin 2 state; thus the amplitude for $\pi^+\pi^+ \to \pi^+\pi^+$ has to be real.

The $\pi\pi$ system is experimentally known to couple strongly at least to the $1^GJ^P = 1^+1^−\rho(765)$ and to the $1^GJ^P = 0^+2^+f^0(1260)$ (Figs. 6 and 7). Thus at least two trajectories contribute in a Regge pole model for $\pi^+\pi^+ \to \pi^+\pi^+$: a negative signature trajectory $\alpha_\rho(t)$ and a positive signature trajectory $\alpha_f(t)$. In addition to $\rho$ and $f$ there have been observed a large number of bumps in mass spectra which may correspond to further particles of higher mass on these trajectories (Fig. 8), but the spin determinations for these particles are either inconclusive [for $\rho_N(1660)$ with $1^GJ^P = 1^+3^-$? Figs. 6 and 7] or non-existent (for $R$, $S$, $T$, $U$, $X$, etc., Fig. 8).

Fig. 6 Distribution in the $\pi^+\pi^−$ mass spectrum for the experiments indicated on the figure.
Fig. 7  As Fig. 6 but for $\pi^\pm\pi^0$ mass spectrum.

Fig. 8  Observed non-strange meson resonances plotted with mass squared versus spin if the spin is determined. If no spin determination exists, just the range in mass squared with probable spin values is shown.
We thus have for the amplitude of $\pi^+\pi^+ \rightarrow \pi^+\pi^+$

$$A(\pi^+\pi^+) = \gamma_{\rho\pi\pi}(t) \frac{1 - e^{-i\pi\alpha_p(t)}}{\sin \pi\alpha_p(t)} \left( \frac{s}{s_0} \right)^{\alpha_p(t)} +$$

$$+ \gamma_{f\pi\pi}(t) \frac{1 + e^{-i\pi\alpha_f(t)}}{\sin \pi\alpha_f(t)} \left( \frac{s}{s_0} \right)^{\alpha_f(t)} \equiv \rho_{\pi\pi} + f_{\pi\pi},$$

(2.1)

where $s_0$ is a scale factor and $\gamma_{\rho\pi\pi}(t)$ and $\gamma_{f\pi\pi}(t)$ are real residue functions, the signs of which are undetermined at this stage. From Eq. (2.1) it is easy to obtain the amplitude for $\pi^-\pi^+ \rightarrow \pi^-\pi^+$; the rule is that if

$$A(ab + 12) = A(+) + A(-),$$

(2.2)

where $A(t)$ corresponds to a Regge pole of signature $T$, then

$$A(\bar{1}b + \bar{a}2) = A(+) - A(-).$$

(2.3)

This rule is, in fact, the definition of signature. The operation involved in going from Eq. (2.2) to Eq. (2.3) is called line reversal; it is equivalent to crossing from the s-channel to the u-channel:

In the language of potential theory (in the t-channel where $t$ is the energy), line reversal corresponds to an exchange force. With this rule the $\pi^-\pi^+$-amplitude is

$$A(\pi^-\pi^+) = - \rho_{\pi\pi} + f_{\pi\pi},$$

(2.4)

where $\rho_{\pi\pi}$ and $f_{\pi\pi}$ denote the two terms in Eq. (2.1).
In order to make Eq. (2.1) real, the two phase factors in \( \gamma_{\rho\pi\pi} \) and \( \gamma_{f\pi\pi} \) in this equation have to be cancelled. This happens clearly if

\[
\gamma_{\rho\pi\pi}(t) = \gamma_{f\pi\pi}(t) = \gamma(t) \tag{2.5}
\]

\[
\alpha_{\rho}(t) = \alpha_{f}(t) = \alpha(t)
\]

since then

\[
A(\pi^+\pi^+) = \frac{2\gamma(t)}{\sin \pi \alpha(t)} \left( \frac{S}{s_0} \right)^{\alpha(t)} \tag{2.6}
\]

\[
A(\pi^-\pi^+) = \frac{2\gamma(t)}{\sin \pi \alpha(t)} e^{i\pi \alpha(t)} \left( \frac{S}{s_0} \right)^{\alpha(t)}
\]

When Eq. (2.5) is satisfied, the \( \rho \) and \( f \) trajectories are said to be exchange degenerate (EXD). A particular consequence of this is that the \( \rho \) residue \( \gamma_{\rho\pi\pi}(t) \) must vanish when \( \alpha_{\rho}(t) = 0 \); to avoid a pole in the amplitude (2.1) at \( \alpha_{f}(t) = 0 \) the residue \( \gamma_{f\pi\pi} \) must vanish at \( \alpha_{f}(t) = 0 \), by EXD (2.5) the same applies to \( \gamma_{\rho\pi\pi} \). Thus the \( \rho \)-exchange amplitude should vanish when \( \alpha_{\rho}(t) = 0 \) or \( t \approx -0.6 \) (GeV/c)\(^2\); this is called "nonsense choosing". Experimentally a good place to test this mechanism is \( \pi^-p \to \pi^0n \), which should be dominated by \( \rho \). In fact, one sees a dip at \( t \approx -0.6 \) (GeV/c)\(^2\). However, there are also other places where this dip mechanism does not work as well.

One should also emphasize the difference between the EXD of trajectories (weak EXD) and the EXD of couplings, too (strong EXD). As we have now proved that \( \alpha_{\rho} = \alpha_{f} \), this holds for all reactions, while the equality of couplings must be shown for each coupling separately. The \( \rho \) and \( f \) need not couple equally to all states, for example their couplings to \( \bar{N}N \) are different.

Here the EXD of \( \rho \) and \( f \) was derived from duality; it can also be understood in terms of the following argument, given prior to duality and explaining the terminology. Consider \( \bar{q}q \) scattering in a potential picture so that \( \rho \) is an \( \ell = 1 \) and \( f \) is an \( \ell = 2 \) resonance. If there are exchange forces present, the potentials in the even and odd parity states are different \([V_{\text{eff}}^\pm(r) = V_{\text{dir}}(r) \pm V_{\text{exchange}}(r) \text{ for } \ell = \text{ even (})^*\text{ and odd } (-)\]].\)
This leads to a separation of the trajectories corresponding to even \( \ell \) and odd \( \ell \) particles. However, in \( \bar{q}q \to \bar{q}q \) (s-channel) the t-channel (corresponding to direct force) contains a \( \bar{q}q \)-state while the u-channel (exchange force) contains a \( q\bar{q} \)-state. Since a \( q\bar{q} \)-state is exotic and experimentally suppressed, the exchange forces are weak. Thus, the even \( \ell \) and odd \( \ell \) trajectories coincide, and are said to be exchange degenerate.

To test the EXD of \( \alpha_\rho(t) \) and \( \alpha_f(t) \) in the region \( t > 0 \), one has to look at the observed particles. Here the EXD is very plausible, but a really conclusive proof is missing. With \( m_\rho^2 = 0.585 \) and \( m_g^2 = 2.77 \), a straight line going through \( \rho \) and \( g \) on the Chew-Frautschi plot has the equation \( \alpha_\rho(t) = 0.465 + 0.91 t \). This predicts a particle at \( \alpha_\rho = 2 \) with mass 1.3 GeV, close to the mass of \( f \). In the region \( t < 0 \) one should, in principle, isolate the trajectories from scattering data and see if they coincide. This, however, is next to impossible to carry out in practice, since one should determine the function \( \alpha(t) \) while, for example, the residue function \( \gamma(t) \) is unknown. It is somewhat simpler to test the fact that Eqs. (2.6) predict that the absolute values of \( \pi^+\pi^- \) and \( \pi^+\pi^- \) amplitudes are equal; we shall return to this later.

In the \( 2^+ \) nonet there is still one particle which could couple to \( \pi^+\pi^- \), the \( f'(1514) \). Since \( m_{f'}^2 = 2.29 >> m_f^2 = 1.60 \), this can certainly not be degenerate with the \( f \). If we add a term \( f'_{\pi\pi} \) to Eq. (2.1) another trajectory of opposite signature would be needed to cancel its contribution to the imaginary part of the amplitude for \( \pi^+\pi^- \to \pi^+\pi^- \), exactly as above the \( \rho \) compensated for the \( f \). However, experimentally there is no candidate for this. For example, G-parity forbids the decay of the \( \Gamma_{\phi(1019)} = 0^{-}1^{-} \phi(1019) \) to \( \pi^+\pi^- \). The only simple way to get around this difficulty is to forbid the coupling of \( f' \) to \( \pi\pi \). Here, this conclusion is a consequence of duality. Experimentally, the coupling of \( f' \) to \( \pi\pi \) is very small (there is no peak at 1514 MeV in Fig. 6).

The decoupling of \( f' \) from \( \pi\pi \) can be given a pictorial formulation in terms of quark diagrams, in which each meson is represented by two lines directed in opposite directions (corresponding to its quark-antiquark content) and each baryon by three parallel lines (three quarks). In the quark model, with the standard nonet mixing angle, the \( f' \) contains only a \( \lambda\bar{\lambda} \)-pair, a \( \pi^+ \) is a \( p\bar{n} \)-pair and a \( K^+ \) is a \( p\bar{\lambda} \)-pair. There is no way of
representing the decay $f' \to \pi^+\pi^-$ by a connected diagram containing quark lines, while this works well for $f' \to K^+K^-:

The fact that $f' \to \pi\pi$ is forbidden can thus be re-expressed by forbidding all disconnected quark diagrams. We shall return to this in connection with duality diagrams.

We emphasize still again that the following assumptions were involved in the derivation of the above results.

i) Analyticity (implicitly).

ii) The Regge pole expansion of the $\pi\pi$ amplitude is dominated by the exchange of two Regge poles.

iii) The imaginary part of the amplitude is dominated by direct channel resonances. Only non-exotic resonances in the sense of quark model exist.

These assumptions express what is called global duality. Nowhere did one assume that the expansions (2.1) and (2.4) represent exactly the $\pi\pi$ amplitude. This is manifestly impossible, since the expressions (2.1) and (2.4) have clearly no poles in $s$ and thus do not contain explicitly the s-channel resonances. In the resonance region the Regge pole amplitudes represent the complete amplitude only in an average sense (see later Fig. 14).

Although the total amplitude $A(\pi^+\pi^-)$ in Eq. (2.6) does not contain resonance poles explicitly, the corresponding partial wave amplitudes $a_\gamma(s)$ vary rapidly and form circles [Schmid circles$^6$] on the Argand plot (complex $a_\gamma$ plane). This is essentially due to the phase factor $e^{-i\pi\alpha(t)}$, which distinguishes $A(\pi^+\pi^-)$ from $A(\pi^+\pi^+)$. Considering only
this factor the partial wave amplitudes \( \tilde{a}_\ell(s) \) are [\( s = 4(q^2 + m_n^2) \)]

\[
\tilde{a}_\ell(s) = \frac{1}{2} \int_{-1}^{1} \, dz \, P_\ell(z) \, e^{-i\pi[\alpha_0 - \alpha' q^2(1-z)]} \]

\[
= i^\ell \, e^{-i\pi \alpha_0} \, e^{i\pi \alpha' q^2} \, j_\ell(-2\pi \alpha' q^2),
\]

(2.7)

because of the relation

\[
j_\ell(z) = (-1)^\ell \frac{1}{2} \int_{-1}^{1} \, dt \, P_\ell(t) \, e^{itz}
\]

(cf. the expansion of \( e^{ikr} \) in spherical waves). In Eq. (2.7), the two first factors give a constant phase and the third factor moves counterclockwise on the complex plane on a unit circle as \( q^2 \) increases, returning to the same value when \( s \rightarrow s + n/\alpha' \), \( n = \pm 1, \pm 2, \ldots \). [Notice that the rotation would be clockwise if we used \( e^{i\pi \alpha(t)} \) instead of \( e^{-i\pi \alpha(t)} \) in the signature factor.] The last factor is a real number determining the absolute value of \( \tilde{a}_\ell(s) \). For \( \ell = 0 \) one has \( j_0(z) = \sin z/z \) and with \( \alpha_0 = \frac{1}{2} \) and \( \lambda = 2q^2 \alpha' \pi \)

\[
\tilde{a}_0(s) = \frac{\sin^2 \lambda}{\lambda} - i \frac{\cos \lambda \sin \lambda}{\lambda}.
\]

On the complex plane this describes a spiral:
and, according to the usual interpretation of circles on the Argand plot as resonances, this spiral corresponds to a set of S-wave resonances with equally spaced masses squared ($\Delta M^2 = 1/\alpha'$). For higher $\ell$ and small masses ("ancestor states"), the circles are very small, since $j_\ell(-2\pi \alpha q^2)$ is zero for $q^2 = 0$ and has its first maximum for $2\pi \alpha q^2 \approx \ell$ or $s \approx \ell$. Thus, the largest circles, or the most strongly coupled resonances, have $s = M^2 \sim \ell$. However, all details are irrelevant, since we have neglected all other $t$-dependent factors but the phase factor $e^{-i\pi \alpha(t)}$, in particular those producing the damping of $A(s,t)$ for large $t$. Even if one takes a successful Regge fit at high energies, extrapolates it to low energies and partial wave analyses, it is hard to reproduce to observed resonance structure in the direct channel in a convincing and stable manner. Thus, all conclusions about Schmid circles are at best only qualitative.

One may also ask how the requirement of a constant (apart from logarithmic terms) $d\theta/dt$ constrains the resonances. The angular distribution of each resonance separately is determined by $P_\ell(\cos \theta)$, where

$$\cos \theta = 1 + \frac{2t}{s - 4m_\pi^2}.$$  \hfill (2.8)

In the neighbourhood of $\theta = 0^\circ$ or $t = 0$, $P_\ell(\cos \theta)$ decreases from one at $\theta = 0^\circ$ the faster, the larger $\ell$ is. To estimate the rate of decrease in terms of $t$ one may use the approximation [remember that $J_0(0) = 1$]

$$P_\ell(\cos \theta) \approx J_0\left(\sqrt{2(1 - \cos \theta)}\right) = J_0\left(\sqrt{-\frac{2\ell}{\sqrt{s - 4m_\pi^2}}}\right),$$

where the last equality follows from Eq. (2.8). Thus at least if the situation is such that at each energy a resonance with $\ell \approx \sqrt{s}$ dominates, the $t$-distribution is constant. This is incompatible with the requirement of resonances lying only on a single linear trajectory ($\ell \approx s$). This incompatibility can be resolved by introducing, in addition to the leading linear trajectory, a set of parallel and lower trajectories (daughter trajectories), so that the most strongly coupled resonances will have $\ell \approx \sqrt{s}$. We shall soon see explicitly how this happens in dual resonance models.
We learned from global duality that the dominant Regge poles in \( \pi \pi \) scattering lie on a single EXD trajectory \( \alpha_p(t) = \alpha_F(t) = \alpha(t) = \alpha_0 + \alpha't \). Consider then the model amplitude\(^7\) (we shall include a normalization factor later)

\[
V(s,t) = \frac{[1 - \alpha(s)] [1 - \alpha(t)]}{[1 - \alpha(s) - \alpha(t)]}, \tag{2.9}
\]

where \( \Gamma(z) \) is the \( \Gamma \) function. We shall show that this is a reasonable model amplitude for \( \pi^+ \pi^- \to \pi^+ \pi^- \) in the sense that

i) it is analytic;

ii) it has Regge asymptotic behaviour for \( s \to \infty \) determined by \( \alpha(t) \);

iii) it has resonance poles (of zero width) at positions determined by \( \alpha(s) \).

By crossing to the \( u \)-channel, in which \( s \) and \( t \) are momentum transfers, \( V(s,t) \) is also the amplitude for \( \pi^+ \pi^- \to \pi^+ \pi^- \). If we relabel the particles so that \( \pi^+ \pi^- \to \pi^+ \pi^+ \) is the \( s \)-channel, the amplitude for \( \pi^+ \pi^- \to \pi^+ \pi^+ \) will be \( V(u,t) \).

The properties of \( V(s,t) \) follow from the properties of \( \Gamma(z) \) given below:

a) \( \Gamma(z) \) is an analytic function the only singularities of which, in the finite complex plane, are poles at \( z = 0, -1, -2, \ldots \), so that when \( z \to -n \)

\[
\Gamma(z) \sim \frac{(-)^n 1}{n!} \frac{1}{z + n}. \tag{2.10}
\]

Fig. 9 shows the behaviour of \( \Gamma(z) \) for real \( z \).

b) \( \Gamma(z) \) satisfies the relations

\[
\Gamma(z + 1) = z \Gamma(z) \tag{2.11}
\]

\[
\Gamma(1 - z) \Gamma(z) = \frac{\pi}{\sin \pi z}. \tag{2.12}
\]

The former generates the poles mentioned in (a) with the residue in expression (2.10).
c) For $|z| \to \infty$, but for $z$ not within the cone $-\pi + \varepsilon < \arg z < \pi - \varepsilon$ ($\varepsilon$ arbitrarily small), around the negative real axis (where $\Gamma$ has poles),

$$
\Gamma(z) \sim \sqrt{2\pi z} \left(\frac{e}{z}\right)^z \left(1 + \frac{1}{12z} + \ldots\right)
$$

(Stirling's formula). For the ratio of two $\Gamma$'s one obtains the simple formula

$$
\frac{\Gamma(z + a)}{\Gamma(z + b)} \sim z^{a-b} \left[1 + \frac{(a - b)(a + b - 1)}{2z} + \ldots\right].
$$

Numerically, formula (2.13) is very accurate and can, in combination with the recursion relations (2.11) and (2.12), be used for the numerical computation of $\Gamma(z)$ for all complex $z$ not too near the negative real axis.
V(s,t) can now immediately be seen to have poles when the two
Γ-functions in the numerator have poles. This happens when \( \alpha(s) = 1, 2, \ldots \)
and when \( \alpha(t) = 1, 2, \ldots \), corresponding to resonances in the s- and
t-channels at masses \( M_n = \sqrt{(n - \alpha_0)/\alpha'} \), \( n = 1, 2, \ldots \). It is also very
easy to find out the residues at the poles by using expression (2.10).
More directly, one may use the following expansion of Euler's B-function,
called here \( B_n \):

\[
B_n(x,y) \equiv \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)} = \sum_{n=0}^{\infty} \binom{n-y}{n} \frac{1}{n + x}
\]

(2.15)

The binomial coefficient is defined by

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{\Gamma(n+1)}{k! \Gamma(n-k+1)} = (-)^k \frac{\Gamma(k-n)}{\Gamma(-n)} = \frac{n(n-1)\ldots(n-k+1)}{k!}
\]

(2.16)

so that it is a polynomial of order \( k \) in \( n \). The result is

\[
V(s,t) = -\sum_{n=1}^{\infty} \frac{\Gamma[n + \alpha(t)]}{\Gamma(n) \Gamma[\alpha(t)]} \cdot \frac{1}{n - \alpha(s)}
\]

\[
= \frac{-\alpha(t)}{1 - \alpha(s)} + \frac{-\alpha(t)[1 + \alpha(t)]}{2 - \alpha(s)} + \frac{-\alpha(t)[1 + \alpha(t)][2 + \alpha(t)]}{3 - \alpha(s)} + \ldots
\]

(2.17)

The expansion (2.15) converges evidently if the residue of the
pole at \( x = -n \) goes to zero when \( n \to \infty \) (\( x \) and \( y \) are fixed). One has,
when \( n \to \infty \):

\[
\binom{n-y}{n} = \frac{(-)^n}{\Gamma(1-y)} \cdot \frac{\Gamma(n+1-y)}{\Gamma(n+1)} \sim \text{const} \ n^{-y}
\]

so that the series converges if \( \text{Re} \ y > 0 \). In Eq. (2.17) this means
that the expansion, in terms of s-channel poles, converges when
\( \alpha(t) < 0 \) or \( t < -\alpha_0/\alpha' \). This region practically contains the s-channel
physical region. An expansion in terms of t-channel poles, obtained
from Eq. (2.17) by interchanging s and t, does not converge in the s-
channel physical region. This is nothing but the well-known fact that
the exchange of a particle of spin J gives in the s-channel the ampli-
tude
\[
\frac{P_j(\cos \theta_s)}{t - m_j^2} \approx \frac{s^j}{t - m_j^2}
\]
which diverges for large \( s \).

The poles in Eq. (2.17) lie on the real axis \([\alpha(s) \text{ is real}]\) and thus correspond to resonances of zero width. The residue of the pole at \( \alpha(s) = \lambda \) is a polynomial of order \( \lambda \) in \( \alpha(t) \) and, since \( \alpha(t) \) is linear in \( t \), it is also a polynomial of order \( \lambda \) in \( t = -2q^2(1 - \cos \theta_s) \) or in \( \cos \theta_s \). According to the partial wave expansion, a pure spin \( \lambda \) particle corresponds to the angular factor \( P_\lambda(\cos \theta_s) \) in the amplitude. Thus, the pole at \( \alpha(s) = \lambda \) contains a sequence of poles of spin \( \lambda \) (the parent) and spins \( \lambda-1, \lambda-2, \ldots, 0 \) (the daughters) (Fig. 10), all with the same mass. There are no ancestors, particles with spin > \( \lambda \).

The complete pole structure of \( V(s,t) \) is shown in Fig. 11 on the \( s,t \) plane. The poles at \( \alpha(s) = n, \alpha(t) = n, n = 1, 2, \ldots \), may coincide in the region \( s > 0, t > 0 \), which is outside the physical region of scattering process \( \pi^+ \pi^- \rightarrow \pi^+ \pi^- \). No double pole appears, however, since the amplitude \( V(s,t) \) vanishes on the lines \( \alpha(s) + \alpha(t) = n, n = 1, 2, \ldots \), so that the double poles of the numerator are reduced to single poles of \( V(s,t) \) only. One of the lines of zeros, \( \alpha(s) + \alpha(t) = 1 \), is redundant in the sense that it is not needed to cancel any double poles. It is there to give a correct power of \( \alpha(t) \) in Eq. (2.17) or to give a correct Regge behaviour.

Figure 11 should be compared with Fig. 12 showing an analogous plot for the experimentally more accessible case of \( \pi N \) scattering. Here the resonances and their parameters are well determined by phase-shift analysis. The similarity between Figs. 11 and 12 is considerable. However, there exists no simple model amplitude which would reproduce the complicated information content of Fig. 12.

Consider then in more detail the behaviour of \( V(s,t) \) near the poles. At the \( \rho \) pole \( (q = \frac{1}{2} \sqrt{s - 4m_\pi^2} \) is the c.m. momentum)

\[
V(s = m_\rho^2, t) \approx -\frac{\alpha(t)}{1 - \alpha(s)} = -\frac{\alpha_0 + \alpha't}{1 - \alpha_0 - \alpha's} = \frac{\alpha_0/\alpha' - 2q^2 + 2q^2 \cos \theta}{s - m_\rho^2} = \frac{a + b \cos \theta}{s - m_\rho^2},
\]

(2.18)
Fig. 10 The low-spin parent and daughter states included in the model amplitude $V(s,t)$.

Fig. 11 The physical regions on the $s,t$ plane of the three different channels of $\pi^+\pi^- \rightarrow \pi^+\pi^-$. The straight lines marked $\alpha(s) = 1$, etc., show the positions of the poles in $s$ and $t$; the dashed lines marked $\alpha(s) + \alpha(t) = 1$, etc., are the lines of zeros of the amplitude. The curve in the region $s > 0$, $t > 0$ is the boundary of the Dalitz plot for $\bar{p}n \rightarrow \pi^+\pi^-\pi^-$. 
Fig. 12 The physical region for $\pi N \rightarrow \pi N$ with the positions of the prominent resonances and the first zeros in their contributions to the spin flip amplitude $B(\pi^-)$ and the spin non-flip amplitude $A'(\pi^-)$ indicated. The first zeros in $B(\pi^-)$ are at $-0.6 < t < -0.4 (\text{GeV/c})^2$, whilst those in $A'(\pi^-)$ are at $t \approx -0.2$.

where in units of GeV$^2$

$$a = \frac{\alpha_0}{\alpha'} - b \approx 0.28$$

$$b = 2q^2 - \frac{1}{2}(m_p^2 - 4m_\pi^2) \approx 0.25 \ .$$

Compare this with what one would obtain from the partial wave expansion

$$f(q, \cos \theta) = \frac{V(s, t)}{8\pi \sqrt{s}} = \sum _\ell \left(2\ell + 1\right) f_{2\ell}(q) P_\ell(\cos \theta) ,$$

in which both the S and P waves resonate at $\sqrt{s} = m_p$. Since only $\ell = 0$ and 1 need be included,

$$\frac{1}{8\pi m_p} V(s = m_p^2, t) = f_0(q) + 3f_1(q) \cos \theta ,$$

where for $\ell = 0, 1$ the partial wave amplitudes are given by the Breit-Wigner forms ($\Gamma$ is the total, $\Gamma^e$ the elastic width)

$$f_{\ell}(q) = \frac{1}{q} \frac{\Gamma_{\ell}^e/2}{m_p - E - i\Gamma_{\ell}/2} \approx \frac{1}{q} \frac{m_p \Gamma_{\ell}^e}{m_p^2 - s - im_p \Gamma_{\ell}} .$$
so that

\[
V(s \approx m_p^2, t) \approx \sqrt{1 - \frac{4m_\pi^2}{m_\rho^2}} \left[ \frac{\Gamma_0^e}{m_\rho^2 - s - i m_\rho \Gamma_0} + \frac{3\Gamma_1^e \cos \theta}{m_\rho^2 - s - i m_\rho \Gamma_1} \right].
\]

(2.20)

The total widths \( \Gamma_\ell \) are equal and zero in Eq. (2.18) and the model is clearly non-unitary, since the sum of the partial widths should give the total width. Considering only the relative importance of the elastic widths one sees that the ratio of the elastic widths to the \( \pi^+\pi^- \) channel of the S- and P-wave resonances has the value \( \Gamma_0^e : \Gamma_1^e = 3a:b \approx 3:1 \). What one actually wants is \( \Gamma(\sigma + \pi\pi) : \Gamma(\rho + \pi\pi) \), where the isospins of \( \rho \) and \( \sigma \) are 1 and 0, respectively. This introduces a factor \( \frac{3}{2} \) [see Eq. (2.36) and remember that \( \sigma + \pi^0\pi^0 \) but \( \rho^0 \not\sim \pi^0\pi^0 \)] and one has \( \Gamma(\sigma + \pi\pi) : \Gamma(\rho + \pi\pi) = 9:2 \).

If \( \Gamma_\rho \approx 120 \text{ MeV} \), this gives \( \Gamma_\sigma \approx 540 \text{ MeV} \). The existence of \( \sigma \) is fairly well established from several independent sources but its parameters are uncertain since it is so broad. In any case \( \Gamma_\sigma \approx 540 \text{ MeV} \) is in the range observed. One piece of evidence for \( \sigma \) is shown in Fig. 13, which shows the angular distribution at the \( \rho \) mass for \( \pi^-p + \pi^-\pi^+n \). This is not \( \pi\pi \) scattering, but the fact that the angular distribution deviates considerably from \( \cos^2 \theta \) (\( \rho \) only) can be associated with the presence of \( \sigma \). Since the deviation is not seen in \( \pi^-p + \pi^-\pi^+p, \sigma \) must have isospin 0.

The angular distribution at the \( \rho \) mass given by Eq. (2.18) is shown in Fig. 14, which should be compared with Fig. 13. Since \( a \approx b \), the S- and P-waves interfere destructively near the backward direction, so as to produce a conspicuous forward peak, but almost nothing in the backward direction. This is in agreement with a Regge pole description of \( \pi^+\pi^- + \pi^+\pi^- \); since there are t-channel (\( \pi^+\pi^- \)) poles but no u-channel (\( \pi^+\pi^+ \)) poles, the reaction should exhibit a forward but no backward peak. Here this description is applied in a very strict sense near a prominent low-energy resonance (local duality). One sees that, in general, a set of resonances are needed to cancel each other in the backward direction and to interfere constructively in the forward direction. Thus, local duality alone predicts the existence of daughters and the model amplitude \( V(s,t) \) only gives a concrete realization of them.
Fig. 13  The distributions in the angle between the initial $\pi^-$ and the final $\pi^-$ in the rest frame of the final state pions at the $\rho$ mass for $\pi^-\pi^+$ and $\pi^-\pi^0$ pion systems.

A similar analysis may be carried out for higher poles; at $s = m_f^2$

$$V(s = m_f^2, t) = \frac{-\alpha(t) - \alpha^2(t)}{2 - \alpha(s)} ,$$

(2.21)

from which one can, as before, infer that the widths of the parent $f^0$ and its daughters $\rho'$ and $\epsilon'$ to the $\pi\pi$ channel are roughly in the ratios

$$\Gamma_{\epsilon'} : \Gamma_{\rho'} : \Gamma_f \approx 0:10:9 .$$
Fig. 14  Angular distributions predicted by the model amplitude (in arbitrary units) at the $\rho$ position ($\alpha_s = 1$) and at the $\phi$ position ($\alpha_s = 2$).

The angular distribution is shown in Fig. 13. The amplitude now has a zero, too (Fig. 11). The existence of $\rho' (J^P = 1^+1^-)$ is more uncertain than that of $\sigma$. In fact, it might manifest itself as a peak in a $\pi^+\pi^0$ mass spectrum at the $f$ mass, while a comparison of Figs. 6 and 7 shows that nothing of the kind is seen. However, one has to keep in mind that there may be clear resonances [like the $N_{1/2}^+(1470)$ in Fig. 3] which do not cause any structure in reaction cross-sections.

So far we have made statements about $V(s,t)$ at a fixed energy neglecting the problem of zero widths. This problem is very deep and at present there is no way of introducing finite-width resonances into the model in a way which would leave the other good properties of the model, as crossing symmetry, unchanged. A complete theory should contain a prescription for the full width of the resonance [$\Gamma_\ell$ in Eq. (2.20)], which determines the width of the Breit-Wigner, and for the partial widths $\Gamma_\ell^i$, which satisfy $\Gamma_\ell = \sum_{\text{all channels}} \Gamma_\ell^i$ and describe the strength of the coupling of the resonance to the different channels it can decay into. All possible
channels have thus to be taken into account, under the unitarity constraint, while our model amplitude only knows about the $\pi\pi$ channel. Introducing correct widths is thus evidently not a formal technical problem, but requires new physical input.

Phenomenologically, however, one may construct different ways of smoothing the $s$-channel poles. The simplest one is to introduce an imaginary part for $\alpha(t)$:

$$\alpha(s) = \alpha_0 + \alpha' s + i \text{Im}\alpha(s),$$  \hspace{1cm} (2.22)

where $\text{Im}\alpha(s) \neq 0$ when $s$ is larger than some threshold $s_0 = (2m_\pi)^2$, above which resonances start to appear. Then near $s = m_R^2$ $\alpha_0 + \alpha' m_R^2 = n$

$$\frac{1}{n - \alpha(s)} = \frac{1}{n - \alpha_0 - \alpha's - i\text{Im}\alpha(s)} = \frac{1/\alpha'}{m_R^2 - s - i(1/\alpha')\text{Im}\alpha(s)},$$  \hspace{1cm} (2.23)

so that comparing with Eq. (2.20) $\text{Im}\alpha(s)$ may be determined by

$$\text{Im}\alpha(m_R^2) = \alpha' m_R \Gamma_R .$$  \hspace{1cm} (2.24)

If $m_R \Gamma_R$ behaves simply when $R$ goes through the different resonances on a trajectory, Eq. (2.24) leads to a convenient parametrization of $\text{Im}\alpha(s)$. Thus for baryon trajectories $m_R \Gamma_R$ increases roughly proportionally to $m_R^2$, and one can write $\text{Im}\alpha(s) = a(s - s_0) \theta(s - s_0)$, where $a$ is a constant. For mesons the situation is more confused. With the parametrization

$$\alpha_p(s) = 0.48 + 0.95 + i 0.46(s - 0.07),$$  \hspace{1cm} (2.25)

corresponding to $\Gamma_p = 120$ MeV, $\Gamma_f = 220$ MeV (experimentally $150 \pm 25$ MeV), Fig. 15 shows the cross-section for $\pi^+\pi^- \rightarrow \pi^+\pi^-$ computed from the formula

$$\sigma(s) = \frac{1}{16\pi \lambda(s, m_\pi^2, m_\pi^2)} \int_{-s + 2m_\pi^2}^{0} dt |V(s,t)|^2 .$$  \hspace{1cm} (2.26)

The same figure shows also the cross-section coming from the first three terms in Eq. (2.17). Within the range of these resonances the results are virtually identical.
The reaction cross-section calculated using the smoothing procedure in Eq. (2.22) with exact $V(s,t)$, with the approximation obtained by taking three first s-channel poles, and with the leading Regge pole term.

The smoothing procedure (2.22) has the following obvious disadvantages: (a) it destroys analyticity and crossing; (b) for general $\text{Im } \alpha(s)$ the pole residue at $\alpha(s) = \lambda$ is no longer a polynomial of order $\lambda$ in $\cos \theta$; (c) it gives the same total width to the parent and to the daughters. In any case, it does not unitarize $V(s,t)$. The way in which $V(s,t)$ violates unitarity even with the smoothing procedure (2.25) can be seen very clearly near the $\rho$ position. Here the zero width $V(s,t)$ gives an angular distribution roughly proportional to $(1 + \cos \theta)^2$, so that the reaction cross-section is

$$\sigma \sim \int_{-1}^{1} dz (1 + z)^2 = 2 + \frac{2}{3}$$
and the S-wave cross-section is three times as large as the P-wave cross-section. However, the partial cross-section \( \sigma_\ell(s) \) for a state of angular momentum \( \ell \) is

\[
\sigma_\ell(s) = (2\ell + 1) \frac{4\pi}{q^2} \sin^2 \delta_\ell \leq (2\ell + 1) \frac{4\pi}{q^2},
\]

indicating that if both \( \ell = 0 \) and \( \ell = 1 \) resonate at the same energy the S-wave cross-section is \( \frac{1}{3} \) of the P-wave cross-section (for the \( \rho \) the unitarity limit is \( 115.8 \text{ mb} \)). Thus the S-wave cross-section as given by \( V(s,t) \) exceeds its unitarity limit roughly by a factor 10. A unitarization method which corrects for this is the K-matrix method\(^8\), which essentially gives a resonance a total width which is equal to its elastic width. As \( \Gamma_0^0 : \Gamma_1^0 \approx 3:1 \), this changes the angular distribution at \( \rho \) to the form \( \left( \frac{1}{2} + \cos \theta \right)^2 \), which leads to \( \sigma_0 \approx \frac{1}{3} \sigma_1 \), in agreement with unitarity and experimental data in Fig. 13. At higher \( \ell \) the resonances can already be inelastic and the violations of unitarity are not as manifest as in connection with the S-wave.

After having treated the low-energy or resonance region in detail, consider next the high-energy behaviour of \( V(s,t) \). Assume that the poles have been displaced away from the real \( s \)-axis as in Eq. (2.22) and use Eq. (2.14). Then

\[
V(s,t) \to \Gamma[1 - \alpha(t)] [1 - \alpha(s)]^a(t) \tag{2.27}
\]

and if \( \text{Im} \alpha(s) \) increases more slowly than \( s \) it can be neglected in formula (2.27), so that in the high-energy limit \((-1 = e^{-i\pi})\)

\[
V(s,t) = \Gamma[1 - \alpha(t)] e^{-i\pi \alpha(t)}(\alpha s)^a(t). \tag{2.28}
\]

This is just the Regge pole term (2.6) with \( s_0 = 1/\alpha' \) and with a uniquely determined residue function -- notice that the \( \text{Im} \alpha(t) \) factor can be reproduced by writing [cf. Eq. (2.12)]

\[
\Gamma[1 - \alpha(t)] = \frac{\pi}{\Gamma[\alpha(t)]} \sin \pi \alpha(t).
\]

Figure 14 shows also the reaction cross-section calculated from Eq. (2.28) by Eq. (2.26). \( V(s,t) \) was also the amplitude for \( \pi^+ \pi^+ - \pi^+ \pi^+ \) in the
u-channel, where s and t are momentum transfers \([\text{since } u = M^2_{\pi^+\pi^+} \text{ and } V(s,t) \text{ has no poles in } u \text{ or no } \pi^+\pi^+\text{-resonances}]\). Let us relabel the particles so that the \(s\)-channel is the \(\pi^+\pi^-\)-channel. Then the amplitude for \(\pi^+\pi^- \rightarrow \pi^+\pi^-\) is \(V(u,t)\) (now there must be no poles in \(s\)) and, since \(u = -s - t + 4m^2_{\pi}\), \(u\) approaches \(-s\) asymptotically and

\[
V(u,t) = \frac{\Gamma[1 - \alpha(t)] \Gamma[1 - \alpha(u)]}{\Gamma[1 - \alpha(t) - \alpha(u)]} \rightarrow \Gamma[1 - \alpha(t)] (\alpha')^\alpha(t).
\]  

(2.29)

Notice that now there is no need to smooth the poles since \(V(u,t)\) has none. Equation (2.29) is just the asymptotic amplitude for \(\pi^+\pi^- \rightarrow \pi^+\pi^-\) given in formula (2.14).

As shown earlier, a constant \(t\)-dependence of \(d\sigma/dt\) requires that the prominent resonances have \(\lambda \sim \sqrt{s}\). The model amplitude \(V(s,t)\) accomplishes this by introducing the daughter trajectories parallel to the leading trajectory, so that the states having the largest elastic width to \(\pi\pi\) have \(\lambda \sim \sqrt{s}\), although all the trajectories themselves have \(\lambda \sim s\).

The reaction \(\pi^+\pi^- \rightarrow \pi^+\pi^-\) was simple since one channel is exotic, and out of three possible terms, \(V(s,t), V(s,u), V(u,t)\), only one appears. In other charge states involving a \(\pi^0\) all channels contain resonances and all three terms appear in certain linear combinations. To determine these it is most economical to use isospin amplitudes (for a treatment of isospin in terms of \(\tau\)-matrices or Cartesian coordinates in isospace, see the lectures by Chan in Ref. 1).

The \(\pi\pi\) system can exist in isostates 0, 1, 2 and one finds, using tables of Clebsch-Gordan coefficients, that

\[
|\pi^+\pi^+\rangle = |22\rangle
\]

\[
|\pi^+\pi^-\rangle = \sqrt{\frac{1}{6}} |20\rangle + \sqrt{\frac{1}{2}} |10\rangle + \sqrt{\frac{1}{3}} |00\rangle
\]

\[
|\pi^+\pi^0\rangle = \sqrt{\frac{1}{2}} |21\rangle + \sqrt{\frac{1}{2}} |11\rangle
\]

\[
|\pi^0\pi^0\rangle = \sqrt{\frac{2}{3}} |20\rangle - \sqrt{\frac{1}{3}} |00\rangle.
\]

(2.30)

where \(|jm\rangle\) is a state with isospin \(j\) and third component \(m\).
Since isospin is conserved, one finds immediately [with the notation $A(\pi^+ \pi^+ \rightarrow \pi^+ \pi^+) = A(++)$, etc., and with s-channel isospin amplitudes $S_0, S_1, S_2$]

$$A(++) = S_2$$

$$A(+-) = \frac{1}{6} S_2 + \frac{1}{2} S_1 + \frac{1}{3} S_0$$

$$A(+)0 = \frac{1}{2} S_2 + \frac{1}{2} S_1$$

$$A(00) = \frac{2}{3} S_2 + \frac{1}{3} S_0 .$$

(2.31)

Earlier we have found that (adding a normalization factor $-\lambda$)

$$S_2 = A(++) = -\lambda V(u,t) .$$

(2.32)

An s-channel isostate with isospin $1$ ($0$) is symmetric (antisymmetric) under the interchange of the final or initial state particles which is equivalent to an interchange of $t$ and $u$ (for example, the Clebsch-Gordan coefficients satisfy $< I_1 I_2 | 1m_1 1m_2 > = (-)^I < I_1 I_2 | 1m_1 1m_1 >$). Thus, s-channel isospin only indicates that

$$S_0 = \beta [V(s,t) + V(s,u)] + \gamma V(t,u)$$

$$S_1 = \alpha [V(s,t) - V(s,u)] .$$

(2.33)

To calculate $\alpha, \beta$ and $\gamma$, we impose the condition that there must be no exotic states in the $t$-channel, i.e. $T_2 = -\lambda V(s,u)$. Thus one has to calculate $T_2$ in terms of $S_1$, or carry out the isospin crossing:
Instead of, for example, \( \pi^+ \pi^- \to \pi^+ \pi^- \) one now groups the particles as \( \pi^+ \pi^- \to \pi^- \pi^- \). Here one has to remember that according to the standard (Condon-Shortley) phase conventions \( |JM| = (-)^M |J - M| \) for integer \( J \), so that \( \pi^- = -\pi^+ \), \( \pi^0 = \pi^0 \). One obtains (we include for reference also the u-channel amplitudes)

\[
A(+) = \frac{1}{6} T_2 + \frac{1}{2} T_1 + \frac{1}{3} T_0 = U_2
\]

\[
A(0^+) = \frac{1}{3} T_2 + \frac{1}{3} T_0 = \frac{1}{2} U_2 + \frac{1}{2} U_1
\]

\[
A(0^-) = \frac{2}{3} T_2 + \frac{1}{3} T_0 = \frac{2}{3} U_2 + \frac{1}{3} U_0
\]  

(2.34)

After elimination of the \( A \)'s the st-crossing matrix \( M_{ts} = M_{ts}^{-1} (T_i = M_{ts}^{1j} S_j) \) is found to be

\[
M_{ts} = \frac{1}{6} \begin{pmatrix}
2 & 6 & 10 \\
2 & 3 & -5 \\
2 & -3 & 1
\end{pmatrix}.
\]

(2.35)

In particular, \( T_2 = \frac{1}{2}(S_2 - 3S_1 + 2S_0) \), and setting this equal to \(-\lambda V(s,u)\) one obtains \( \alpha = -\lambda \), \( \beta = -\frac{3}{2}\lambda \), \( \gamma = \lambda/2 \). If we absorb the normalization factor \(-\lambda\) to the definition of \( V(s,t) \), we have

\[
S_0 = \frac{3}{2}[V(s,t) + V(s,u)] - \frac{1}{2}V(t,u)
\]

\[
S_1 = V(s,t) - V(s,u)
\]

(2.36)

\[
S_2 = V(t,u)
\]

The t-channel amplitudes \( T_0 \), \( T_1 \) and \( T_2 \) are obtained from Eq. (2.36) by the transformation \( s \leftrightarrow t, u \) unchanged. By combining Eqs. (2.36) with (2.31) the amplitudes for different charge states can be written down, for instance

\[
A(\pi^+ \pi^0) = \frac{1}{2}[V(s,t) + V(u,t) - V(s,u)]
\]

\[
A(\pi^0 \pi^0) = \frac{1}{2}[V(s,t) + V(u,t) + V(s,u)]
\]  

(2.37)
A(π⁺π⁰) is invariant under the interchange of s and u and A(π⁰π⁰) is invariant under all permutations of s, t, u, in agreement with crossing symmetry.

The asymptotic behaviour in the s-channel for s → ∞ and t small and fixed of the amplitudes Tᵢ with definite t-channel isospin is given by

\[ T_0 \approx \frac{3}{2} \frac{\Gamma[1 - \alpha(t)]}{\Gamma[1 + \alpha(t)]} \left[ 1 + e^{-i\pi\alpha(t)} \right] (\alpha's) \alpha(t) \]

\[ T_1 \approx \frac{\Gamma[1 - \alpha(t)]}{\Gamma[1 - \alpha(t)]} \left[ 1 - e^{-i\pi\alpha(t)} \right] (\alpha's) \alpha(t) \]

\[ T_2 = V(s,u) \approx \frac{1}{\Gamma(\alpha't)} \frac{1}{\cos(\pi\alpha's)} \to 0. \]

T₂, which corresponds to an exotic exchange in the t-channel, vanishes faster than any power of s for s → ∞ at fixed t.

Notice that for the derivation of T₂ → 0, it is crucial that the slopes in α(s) and α(t) are equal, which was assumed to be the case earlier.

We shall conclude this section by a list of some additional topics relevant to ππ scattering.

a) **Satellites**

The expression (2.9) for V(s,t) can be modified without destroying any of its basic properties by adding to it satellite terms, which for s → ∞ behave as (α's)⁻ⁿ⁻², n > 0 and which do not have poles on the parent trajectory. Thus, for example, the term

\[ \frac{\Gamma[1 - \alpha(s)] \Gamma[1 - \alpha(t)]}{\Gamma[2 - \alpha(t) - \alpha(s)]} = Bₐ[1 - \alpha(s), 1 - \alpha(t)] \]

(2.38)

behaves as (α's)⁻¹, when s → ∞ and the residue at the first pole, α(s) = 1, is just a constant implying that Eq. (2.38) has at the ρ mass only a resonance of spin 0. Similarly, at α(s) = ², Eq. (2.38) contains particles of spins 0, 1, ..., ² - 1. The satellite ambiguity is always present, but it only affects conclusions on the daughter level.
b) Adler zero

The model amplitude \( V(s,t) \) was obtained from Eq. (2.38) by multiplying by \( 1 - \alpha(s) - \alpha(t) \), which is needed to obtain the correct high-energy behaviour and correct spin structure. Thus \( V(s,t) \) vanishes on the line \( \alpha(s) + \alpha(t) = 1 \), in addition to vanishing on the lines \( \alpha(s) + \alpha(t) = 2, 3, \ldots \), coming from \( B_\pi \). Alternatively, the Adler condition states that the \( \pi\pi \) scattering amplitude has to vanish when one of the four-momentum vectors of the pions goes to zero, i.e., when \( s = t = u = m_\pi^2 \). This zero coincides with the extra line of zeros if \( \alpha(m_\pi^2) = \frac{1}{2} \), i.e., if \( \alpha(t) = \frac{1}{2} + \alpha'(t - m_\pi^2) \).

c) Normalization

The normalization factor \(-\lambda\) of the \( \pi\pi \) scattering amplitude can be related to the coupling constant \( g_{\rho\pi\pi} \) or to the \( \rho \) width. This is done by computing the matrix element for the Feynman diagram with only \( \rho \) in the \( s \)-channel and with the \( \rho\pi\pi \) vertex function defined as \( G_\rho(p_a - p_b) \mu \), corresponding to

\[
G_\rho^2 = \frac{1}{3} \frac{G_\rho^2}{16\pi m_\rho^2} (m_\rho^2 - 4m_\pi^2)^{3/2} \approx 120 \text{ MeV}.
\]

The matrix element comes out to be

\[
G_\rho^2 \frac{t - u}{s - m_\rho^2 - i\epsilon}.
\]

and, when this is compared with Eq. (2.18), one finds that \( \lambda = 2G_\rho^2 \approx 50 \).

This determination is the simplest case of "bootstrap consistency" or factorization, stating more generally in fairly loose terms that an \( n \)-point function can be divided in two parts with \( k \) and \( n - k \) particles, \((k = 2, \ldots, n - 2)\), so that both parts are good \( k \) and \((n - k)\) point functions.

d) Scattering lengths

An S-wave scattering length is defined by the effective range expansion

\[
q \cot \delta_0^I = \frac{1}{a_0^I} + r_i^2 q^2 + \ldots.
\]
where q is the c.m. momentum and $a_0^I$ is the scattering length in the isos- state I = 0, 1, 2. Since the partial wave amplitude is in the elastic region given by

$$f_0^I(q) = \frac{1}{q} \cdot \frac{1}{\cot \delta_0^I - i},$$

the reaction cross-section near threshold ($q \approx 0$), where only the S-wave contributes, is given by

$$\sigma^I(q \approx 0) = 4\pi (a_0^I)^2.$$

The scattering length thus essentially determines the strength of interaction near threshold.

The $\pi\pi$ scattering lengths can now immediately be calculated from the formulae (2.36) by taking into account the normalization convention

$$f_0^I(\theta) = \frac{1}{8\pi \sqrt{s}} S_I(s, t).$$

This gives at threshold, $s = 4m_\pi^2$, $t = u = 0$,

$$a_0^I = \frac{1}{16\pi m_\pi} S_I(s = 4m_\pi^2, t = 0),$$

which can be easily evaluated using equations appearing earlier.
3. **FURTHER TWO-BODY REACTIONS, DUALITY DIAGRAMS**

Consider first KK scattering in the charge states $K^+K^+ \rightarrow K^+K^+$ and $K^+K^0 \rightarrow K^+K^0$. Now both isostates with $I = 0$ and 1 are exotic since they have $Y = 2$; in $\pi \pi$ scattering only $I = 2$ was exotic. We shall derive some results on the couplings of the different trajectories, which then will be shown to follow at once from duality diagrams.

Since the $K\bar{K}$ system does not have definite G-parity, many meson states can couple to it. The most strongly coupled ones are in the $1^-$ nonet the $\rho$, $\omega$ and $\phi$, and in the $2^+$ nonet the $A_2$, $f$, and $f'$. Further weakly coupled states are neglected. Then the Regge pole amplitude will be a sum of terms of the form

$$\rho_{KK} = \gamma_{\rho K+\bar{K}^+}(t) \gamma_{\rho K+\bar{K}^0}(t) \frac{1 - e^{i\pi\rho(t)}}{\sin \pi\alpha(t)} \left(\frac{s}{s_0}\right)^{\rho(t)},$$

similarly for the other mesons. The sign of Eq. (3.1) is undetermined at the present stage; it is only important to relate $K^+K^+ \rightarrow K^+K^+$ and $K^+K^0 \rightarrow K^+K^0$ correctly. This is done by relating the $MK^+K^+$ vertex to the $MK^+\bar{K}^0$ vertex:

$$K^+ \quad K^+ \quad \text{M} \quad I=0,1$$

by using SU(2) or SU(3) Clebsch-Gordan tables. One finds that changing $K^+$ to $K^0$ involves a sign change, if the isospin of $M$ is 1, and no change if the isospin of $M$ is 0. Thus, choosing positive signs for $\Lambda(K^+K^+)$,

$$\Lambda(K^+K^+) = \rho_{KK} + \omega_{KK} + \phi_{KK} + \Lambda_{KK} + f_{KK} + f'_{KK},$$

$$\Lambda(K^+K^0) = -\rho_{KK} + \omega_{KK} + \phi_{KK} - \Lambda_{KK} + f_{KK} + f'_{KK}.$$  

Since both $s$-channels are exotic, both these amplitudes have to be real. This happens only if simultaneously $\rho$ and $A_2$, $\omega$ and $f$, $\phi$ and $f'$ cancel
each other in the imaginary part, i.e., are pairwise exchange degenerate (any cancellation between \( \omega \) and \( f' \), say, is experimentally out of the question). Thus KK \( \rightarrow \) KK leads to EXD of trajectories with same isospin but opposite G-parity and signature; \( \pi \pi \) scattering led to EXD of trajectories with different isospin, opposite signature, and same G-parity.

Imposing the EXD of trajectories and residues one finds

\[
A(K^+K^+) = \frac{2\gamma_{fKK}^2(t)}{\sin \pi \alpha_f(t)} \left( \frac{s}{s_0} \right)^{\alpha_f(t)} + \frac{2\gamma_{\rho KK}^2(t)}{\sin \pi \alpha_\rho(t)} \left( \frac{s}{s_0} \right)^{\alpha_\rho(t)} + \frac{2\gamma_{\phi KK}^2(t)}{\sin \pi \alpha_\phi(t)} \left( \frac{s}{s_0} \right)^{\alpha_\phi(t)},
\]

while \( A(K^+K^0) \) is the same except for the sign of the \( \rho \)-term.

Finally, consider \( K^+\pi^+ \rightarrow K^+\pi^+ \), in which the contributing trajectories in the t-channel are \( f \) and \( \rho \) with the residues \( \gamma_{fKK}(t) \gamma_{f\pi\pi}(t) \) and \( \gamma_{\rho KK}(t) \gamma_{\rho\pi\pi}(t) \). Since again \( K^+\pi^+ \) is an exotic state (\( T = \frac{3}{2} \) meson), the amplitude has to be real and one finds as before that \( f \) and \( \rho \) are EXD and as a new piece of information that their couplings to KK are equal:

\( \gamma_{fKK}(t) = \gamma_{\rho KK}(t) \). Inserting this in Eq. (3.3) only the \( \phi \) contribution remains in \( A(K^+K^0) \):

\[
A(K^+K^+) = \frac{4\gamma_{\rho KK}^2(t)}{\sin \pi \alpha_\rho(t)} \left( \frac{s}{s_0} \right)^{\alpha_\rho(t)} + \frac{2\gamma_{\phi KK}^2(t)}{\sin \pi \alpha_\phi(t)} \left( \frac{s}{s_0} \right)^{\alpha_\phi(t)}
\]

\[
A(K^+K^0) = \frac{2\gamma_{\phi KK}^2(t)}{\sin \pi \alpha_\phi(t)} \left( \frac{s}{s_0} \right)^{\alpha_\phi(t)}.
\]

There is no need or possibility to go further; the duality framework allows \( \alpha_\rho(t) \) and \( \alpha_\phi(t) \) to be non-degenerate in agreement with the experimentally observed SU(3) breaking. Altogether there will be three vector-meson trajectories; in addition to the previously mentioned \( \rho (= \text{EXD } \rho, A_2, \omega, f \text{ trajectory}) \) and \( \phi (= \text{EXD } \phi, f' \text{ trajectory}) \) one has the \( K^* (= \text{EXD } K^*, K^{**} \text{ trajectory}) \) contributing, for example, to \( K\pi \rightarrow K\pi \).

The trajectories are equally spaced roughly by \( m_K^2 = \gamma_4 \):

\[
\alpha_\rho(t) \approx \frac{1}{2} + t
\]

\[
\alpha_{K^*}(t) \approx \frac{1}{4} + t
\]

\[
\alpha_\phi(t) \approx \frac{1}{2} + t
\]
The amplitudes for the line-reversed reactions $K^- K^+ \rightarrow K^- K^+$ and $K^- K^0 \rightarrow K^- K^0$ are obtained from Eq. (3.2) by changing the signs of odd signature terms (Section 2) and, accordingly, from Eqs. (3.4) by multiplying the two terms by $e^{-i\pi\alpha_P(t)}$ or $e^{-i\pi\alpha_R(t)}$, respectively [cf. Eq. (2.6)].

The decoupling of non-strange exchanges in $A(K^+ K^0)$ in the duality framework is not an immediately evident result and was obtained here after some calculation. What happens, in general, is most easily seen by using duality diagrams\textsuperscript{9,10}. These can be drawn for arbitrary numbers of particles and the rules and terminology are as follows:

i) Each meson is represented by two quark lines running in opposite directions and each baryon by three parallel quark lines (since $M = q\bar{q}$ and $B = qqq$ in the quark model).

ii) Disconnected graphs in which a particle does not exchange quark lines with others are excluded. These graphs are of the types (for meson-meson scattering)

\begin{center}
\begin{tikzpicture}
\draw[->, thick] (0,0) -- (1,0);
\draw[->, thick] (0,1) -- (1,1);
\end{tikzpicture}
\hspace{1cm}
or
\begin{tikzpicture}
\draw[->, thick] (0,0) to [out=90, in=0] (1,1);
\end{tikzpicture}
\hspace{1cm}
\begin{tikzpicture}
\draw[->, thick] (0,0) -- (1,0);
\draw[->, thick] (0,1) -- (1,1);
\end{tikzpicture}
\end{center}

The former is excluded since it corresponds to the background-Pomeron part, the latter is excluded since, for example, the coupling of $f'$ to $\pi\pi$ is forbidden in the duality framework.

iii) The connected graphs are either planar or non-planar if their quark lines do not cross or do cross, respectively. In a non-planar graph there is always a two-particle channel, which contains quarks in combinations other than $q\bar{q}$ or $qqq$. These are exotic channels and thus have real amplitudes in the duality framework. A planar graph may also contain exotic channels (an illegal planar graph).

Applying these rules to $K^+ K^+$ scattering one finds that the connected graphs are
and the one obtained from this by interchanging the p's and λ's. These are non-planar and correspond to real amplitudes, which, of course, was already known, since $K^+K^+$ is exotic. By line reversal or $s \leftrightarrow u$ crossing one may go to $K^-K^+ \rightarrow K^-K^+$ and unfold the intersection of the quark lines:

This introduces a factor $e^{-i\pi\alpha(t)}$ in the Regge pole amplitude, as was found earlier. On the other hand, for $K^-K^0 \rightarrow K^-K^0$ only one diagram is possible, $\bar{p}n$ in the t-channel being excluded since there is no charge exchange:

In the last diagram one has the $\phi$ Regge pole in the t-channel [as in Eqs. (3.41)] dual to the $\rho$ in the s-channel. In the Regge limit this is just our previous result, stating that only $\phi$ exchange contributes to $K^-K^0 \rightarrow K^-K^0$, now derived almost trivially by duality diagrams.
The duality properties expressed by the duality diagrams can be made more explicit by constructing the Veneziano amplitudes corresponding to these diagrams. As such, the duality diagrams say nothing about space-time properties. The rule is just to attach a function \((M,M' = \rho, k^*, \phi)\)

\[
V_{MM'}(s,t) = \frac{\Gamma[1 - \alpha_M(s)] \Gamma[1 - \alpha_M'(t)]}{\Gamma[1 - \alpha_M(s) - \alpha_M'(t)]}
\]

(3.6)

to each duality diagram. The sign is positive, if no minus signs appear in the quark model wave functions of the external particles. For \(0^-\) mesons only \(\pi^0\) and \(\eta\) have minus signs and thus

\[
A(K^-\kappa^+) = \beta [V_{\rho\rho}(t,s) + V_{\rho\phi}(s,t)]
\]

\[
A(K^-\kappa^0) = \beta V_{\rho\phi}(s,t)
\]

(3.7)

where \(\beta\) is a normalization parameter. The duality diagrams show immediately that terms like \(V_{\phi\phi}(s,t)\) or \(V_{\rho\rho}(s,t)\), which in general are allowed, do not appear within the duality framework. When \(s \to \infty\) the limits of Eqs. (3.7) coincide with those obtained from Eqs. (3.4), if \(s_\theta = 1/\alpha'\) and if the coupling constants of \(\rho\) to \(\kappa\kappa\) and \(\phi\) to \(\kappa\kappa\) satisfy \(g_{\rho\kappa\kappa}^2 = g_{\phi\kappa\kappa}^2\).

Consider then meson-baryon (MB) scattering, in particular, \(KN \to KN\). Since there is one exotic channel, \(KN\) with \(Y = 2\), there is only one legal duality diagram:

```
\( \begin{array}{c}
\text{K}^- \\
\lambda \\
\text{p} \\
\text{K}^- \\
\text{p} \\
\text{n} \\
\text{p} \\
\end{array} \)
```

This type of diagram is the only legal MB diagram with meson exchange. The fact that the \(K^+p\) channel (defined now as the s-channel) is exotic leads again to EXD of contributing Regge poles, but now the t-channel \((K^+K^-)\) and the u-channel \((K^-p)\) are separate and are separately constrained.
by $\text{Im } A(\text{KN}) = 0$. The conclusions about the $t$-channel Regge poles $(\rho, \omega, \Lambda_2, f, f', \phi)$ are essentially as in KK scattering. Notice, however, that now $\gamma_{\text{FKK}}^2 \rightarrow \gamma_{\text{FKK}}^2 \gamma_{\text{NN}}^2$ and, since the $\rho$ and the $\omega$ couple differently to $\bar{N}N$, one cannot go from Eq. (3.3) to Eqs. (3.4) for $K^+ p$ scattering. Also, one finds that $\gamma_{\phi NN} = 0$, in agreement with the fact that in the only allowed duality diagram shown above there is no $\lambda \bar{\lambda}$ in the $t$-channel. New results are obtained by requiring that the $u$-channel ($K^- p$) Regge poles lead to a real $K^+ p$ amplitude (backward scattering):

$\begin{align*}
K^+ & \longrightarrow \Lambda, \Sigma \\
p & \longrightarrow K^+
\end{align*}$

To get an idea of the states coupling to $K^- p$, see Fig. 4. There the first bump occurs at $M_{K^- p} \approx 1800$ MeV, the second at $M_{K^- p} \approx 2100$ MeV. Similarly, Fig. 16 shows a clear $K^- p$ resonance at $M_{K^- p} \approx 1520$ MeV in a production experiment. A detailed analysis then shows that the following states couple strongly to $K^- p$ (Figs. 17 and 18):

$I = 1 \quad \tau = + \Lambda_{3/2+}^{(1385)}, \Lambda_{1/2+}^{(2030)}, \ldots = \Sigma_3^{(+\rho)}$\]

$I = 0 \quad \tau = + \Lambda_{1/2+}^{(1115)}, \Lambda_{3/2+}^{(1815)}, \ldots = \Lambda_3^{(e^+)}$

where the notation is $\Lambda_{3/2}^0 (\text{Mass})$ for $I = 0$, $\Sigma$ for $I = 1$. For baryons coupling to $0^{-1/2}^+$ the signature is $\tau = +$ for a trajectory with particles $1/2, 5/2, \ldots$; $\tau = -$ for $3/2, 7/2, \ldots$; the indices $\alpha, \beta, \gamma$ and $\delta$ refer to trajectories starting at $1/2^+, 1/2^-, 3/2^-, 3/2^+$, respectively (there is experimentally no $1/2^-$ particle on $\Sigma_\beta$). Exactly as for $t$-channel trajectories, one can now show that the amplitude for $K^+ p$ is real if $\Lambda_{\alpha}$ and $\Lambda_{\gamma}$ are EXD and if $\Sigma_{\beta}$ and $\Sigma_\delta$ are EXD. However, there is no way of proving that $\Lambda$ and $\Sigma$ should be EXD, while in the meson case all four trajectories, $\rho, A_2, f, \omega$ were EXD. These EXD properties are very well satisfied experimentally (Figs. 17 and 18) and this is probably the best evidence for EXD. A
closer analysis shows that also the couplings are EXD. In \( \pi N \) scattering there is no exotic channel and, as a consequence of this, EXD does not work as well for \( N \) and \( \Delta \) trajectories.

When the direct-channel state is exotic it is trivially impossible to draw a planar duality diagram and duality diagrams give nothing new. However, there are cases in which the \( s \)-channel is non-exotic but a planar duality cannot be drawn. These are for meson-baryon scattering

\[
\begin{align*}
K^- B & \rightarrow \pi^- B' \\
\pi^+ B & \rightarrow K^0 B' \\
K^+ B & \rightarrow K^0 B' \\
\bar{K} B & \rightarrow M^0 B'
\end{align*}
\]  

(3.8)
Fig. 17  The established $I = 0$ $\bar{K}N$ resonances plotted with mass squared versus spin.

Fig. 18  The established $I = 1$ $\bar{K}N$ resonances plotted with mass squared versus spin.
where $B$ and $B'$ are any non-exotic baryons and $M$ is a meson which does not contain $\lambda$. For instance, for $K^- p + \pi^- \Sigma^+$ the $\lambda$ quark in $K^-$ has to get to the $\Sigma^+$ and only a non-planar forward diagram can be drawn:

$$
\begin{array}{c}
K^- \quad \lambda \\
\downarrow \quad \nu \\
p \quad \pi^- \\
\downarrow \\
\Sigma^+ \quad \nu
\end{array}
$$

However, it is possible to draw the planar diagram

$$
\begin{array}{c}
K^- \quad \lambda \\
\downarrow \\
p \quad \pi^- \\
\downarrow \\
\Sigma^+ \quad \nu
\end{array}
$$

This type of diagram is the only legal $MB$ diagram with baryon exchange. As a consequence, the forward amplitude for $K^- p + \pi^- \Sigma^+$ should be real, while the backward amplitude will have an imaginary part. The direct-channel resonances will reproduce this situation if they appear with alternating signs in the forward direction, while averaging out to something non-zero in the backward direction. This, in fact, is just what happens experimentally in the resonance region.

This result can also be understood in terms of $Regge$ poles and global duality as follows. At high energies, $K^- p + \pi^- \Sigma^+$ is presumably dominated by the exchange of $K^*(890)$ and $K^{**}(1420)$ Regge poles. These particles are the strange members of the $1^-$ and $2^+$ nonets and as the $\rho$ and $A_2$ were shown to be EXD one can show that $K^*$ and $K^{**}$ are EXD, for example, by demanding the amplitude for $K^+ \pi^+ + \pi^+ \Sigma^+$ to be real [cf. Eq. (3.5)]. The same Regge poles dominate also the line reversed reaction $\pi^+ p \rightarrow K^+ \Sigma^+$, for which a planar duality diagram can be drawn:
Thus, duality diagrams show that the two EXD poles $K^*$ and $K^{**}$ appear in such a combination that the amplitude is real for $K^- p \to \pi^- \Sigma^+$ and complex for $\pi^+ p \to K^+ \Sigma^+$, exactly as $\rho$ and $f$ cancel in the imaginary part for $\pi^+ \pi^- \to \pi^+ \pi^-$, but produce an imaginary part for $\pi^+ \pi^- \to \pi^+ \pi^-$. In analogy with Eq. (2.6) the amplitude for $\pi^+ p \to K^+ \Sigma^+$ is obtained from the real amplitude of $K^- p \to \pi^- \Sigma^+$ by multiplying by the phase factor $e^{-\pi \alpha_{K^*}(t)}$. Thus their absolute values are the same and one predicts that, in the limit in which only the two EXD Regge poles $K^*$ and $K^{**}$ contribute,

$$\frac{d\sigma}{dt}(K^- p \to \pi^- \Sigma^+) = \frac{d\sigma}{dt}(\pi^+ p \to K^+ \Sigma^+).$$\hspace{1cm}(3.9)$$

Comparison with experiment\textsuperscript{11) shows that $K^- p \to \pi^- \Sigma^+$ cross-section is larger by a factor two. One may devise more analogous tests\textsuperscript{11) for pairs of line-reversed reactions, and it is generally observed that the cross-section corresponding to the amplitude predicted to be real tends to be experimentally larger than the line-reversed cross-section. This is a clear evidence against a model in which only the two EXD Regge poles $K^*$ and $K^{**}$ contribute, but there are also many natural reasons for the discrepancy: subsidiary trajectories, violations of EXD, Regge cuts, etc.

If duality diagrams are applied to baryon-baryon (BB) scattering, one finds that the only planar one
has in the s-channel an exotic $qar{q}qar{q}$ state. If, for example, pp elastic scattering is analysed in terms of Regge poles, one finds that, in addition to the dominant Pomeron contribution, there is a sizeable contribution from the exchange of the $I = 0$ $f$ and $\omega$ Regge poles (which are EXD to make the non-Pomeron part of the pp amplitude real). The presence of these poles manifests itself as a decrease of the pp total and elastic cross-sections when energy increases. Duality diagrams show now that these non-exotic Regge poles should, in $BB$ scattering, be dual to some exotic resonances in the $\bar{B}B$ channel. This is a rather surprising result, since a basic rule of the duality framework is that exotic amplitudes should vanish, while here they clearly are non-zero. There are several ways out of this difficulty, for example, one may suggest that there really exist exotic resonances which couple only to $\bar{B}B$. However, at present it is not possible to make an experimental distinction between various theoretical proposals.

Summarizing, the duality diagrams can be conveniently used to determine:

a) whether an amplitude is real or complex within the duality framework;
b) what in one channel is dual to what in another channel;
c) what dual model amplitudes contribute.

As such, the duality diagrams make statements only about the internal symmetry properties [SU(3) broken in the quark model way] of the particles involved and nothing about the space-time properties. Thus, the question whether, for example, in the meson-baryon case duality diagrams apply to spin non-flip or flip amplitudes has no definite answer, but has to be answered empirically. In fact, it seems that duality works better for spin flip than spin non-flip amplitudes, for which the corrections due to cuts, etc., tend to be larger. An example of this is $\pi^- p \rightarrow \pi^0 n$, in which the nonsense zero expected in the Regge pole terms at $t \approx -0.6$ (GeV/c)$^2$ is seen (Fig. 12) to be given by the resonances at $t = -0.2$ for the spin non-flip and at $t = -0.6$ for the spin flip amplitude. Thus, one may also expect the EXD prediction (3.9) to be more valid for the spin non-flip amplitude, if only this could be isolated from experimental data.
4. APPLICATION TO $\bar{p}n \rightarrow \pi^+ \pi^- \pi^-$

In the kinematical diagram for $\pi\pi$ scattering (Fig. 11), the region $s > 0$, $t > 0$ is inaccessible in a scattering experiment. However, this region is reached physically if one gives the particle $p_b$ so large a mass that the decay process $p_b \rightarrow \bar{p}_a + p_1 + p_2$ is possible:

Then $s = (p_1 + p_2)^2$ and $t = (\bar{p}_a + p_1)^2$ can vary within the Dalitz plot on the $s,t$ plane, drawn in Fig. 11 for $m_b = 2\Lambda$, in anticipation of the treatment of $\bar{p}n \rightarrow \pi^+ \pi^- \pi^-$. If one assumes that $V(s,t)$, in spite of the change in the mass of $p_b$, still describes the dynamics of the decay process $p_b \rightarrow \bar{p}_a + p_1 + p_2$, then this can be used to probe the structure of $V(s,t)$ in the region $s > 0$, $t > 0$.

Consider the annihilation process $\bar{p}n \rightarrow \pi^+ \pi^- \pi^-$ at rest. Since the $\bar{p}$ is annihilated at rest the initial state has orbital angular momentum $L = 0$ and the total angular momentum $J$ equals $S$, where either $S = 0$ (singlet) or $S = 1$ (triplet). Since the initial state is charged, it has clearly $I = 1$. For a nucleon-antinucleon system $P = (-)^{L+1} = -1$, since $L = 0$. Thus, the initial state is either $J^P = 0^-$ or $J^P = 1^-$, but since $G = (-)^{L+S+I}$ the latter state has $G = (-)^{0+1+1} = +$ and cannot decay into three pions. The initial state $\bar{p}n$ is thus uniquely an $I^GJ^P = 1^-0^-$ state, or has exactly the same quantum numbers as the pion. Thus one may try to employ a matrix element for $\pi\pi$ scattering for the decay process $\bar{p}n \rightarrow \pi^+ \pi^- \pi^-$ by just changing the mass of $p_b$. Notice that $m_b$ does not appear explicitly in our dual matrix element, it just appears in the kinematics in the relation $s + t + u = m_b^2 + 3m_n^2$.

Notice that $\bar{p}p \rightarrow \pi^+ \pi^- \pi^0$ has in the initial state both $I = 0$ and 1. Closer analysis shows that the initial state at rest can have the quantum numbers of either $\pi$ or $\omega$. The analysis is thus more complicated. Also, since the final state has no exotic $\pi\pi$ channels, it is less interesting than $\bar{p}n \rightarrow \pi^+ \pi^- \pi^-$, where there are no resonances in the $\pi\pi\pi^-$-system.
We shall now assume that the amplitude $A(s,t)$ for $\bar{p}n + \pi^+\pi^-\pi^-$ is similar to the one we used in analysing $\pi^+\pi^-$ scattering; a closer analysis of data shows that the fit given by the leading terms $V(s,t)$ [Eq. (2.9)] is considerably improved if one includes a satellite term [Eq. (2.38)]:

$$A(s,t) = -\beta \frac{\Gamma(1 - \alpha(s))}{\Gamma(1 - \alpha(s) - \alpha(t))} + \gamma \frac{\Gamma(1 - \alpha(s))}{\Gamma(2 - \alpha(s) - \alpha(t))} =$$

$$= \left\{ \gamma - \beta \left[ 1 - \alpha(s) - \alpha(t) \right] \right\} B\left[ 1 - \alpha(s), 1 - \alpha(t) \right].$$

(4.1)

The distribution of events on the $s = M_{\pi^+\pi^-}^2$ versus $t = M_{\pi^+\pi^-}^2$ Dalitz plot (normally there is no way to distinguish the two $\pi^-$'s and each event is plotted twice) is then given by

$$\frac{\partial^2 \sigma}{\partial s \partial t} \propto |A(s,t)|^2.$$

(4.2)

(Remember that the phase space distribution on the Dalitz plot is constant.) For any comparison with experimental data one again has to remove the poles in $\alpha(s)$ and $\alpha(t)$ away from the real axis by giving the trajectory an imaginary part as in Eq. (2.25).

A look at the experimental data (Fig. 19; the other half of the Dalitz plot is the same as the one displayed) shows immediately that the experimental distribution is very much similar to the one expected on the basis of Fig. 11. One sees clearly the concentration of events at $\alpha(s) = 1$ (p band) or $\alpha(s) = 2$ (f band) and at $\alpha(t) = 1$ or $\alpha(t) = 2$ and the hole in the middle corresponding to the line $\alpha(s) + \alpha(t) = 3$. In fact, this hole is so deep and the depletion of events on the lines $\alpha(s) + \alpha(t) = 2$ or 4 is so much weaker that a fit to the data will require an additional line of zeros at $\alpha(s) + \alpha(t) = 3$. This is obtained by setting $\beta = -\gamma/2$ in Eq. (4.1) (then $A(s,t) = [\alpha(s) + \alpha(t) - 3] B\left[ 1 - \alpha(s), 1 - \alpha(t) \right]$) and explains qualitatively why a satellite term is needed. The fit to the Dalitz plot originally obtained by Lovelace with $\beta = 0$ in Eq. (4.1) is shown in Fig. 19; Fig. 20 gives the distribution in the $\pi^-\pi^-$ mass and shows how the inclusion of the two terms in (4.1) improves the fit.

The structure of the Dalitz plot, following from $A(s,t)$, can be made very obvious by expanding simultaneously in poles in $\alpha(s)$ and $\alpha(t)$. We
Fig. 19  The experimental Dalitz plot [from P. Aminos et al., Phys. Rev. Letters 20, 402 (1968)] for $\bar{p}n \rightarrow \pi^+\pi^-\pi^-$ at rest and the theoretical distribution from Eq. (4.1) with $\beta = 0$. Since there are two $\pi^-$'s, each event is plotted twice and the Dalitz plot is symmetric around the diagonal.

Fig. 20  The $\pi^-\pi^-$ mass spectrum for $\bar{p}n \rightarrow \pi^+\pi^-\pi^-$ with two theoretical curves [see Eq. (4.1)].
have earlier given an expansion of $V(s,t)$ in terms of $s$-channel poles [Eq. (2.17)], which converged in the $s$-channel scattering region; by symmetry one obtains an expansion in the $t$-channel poles. Neither of these converges in the region $s > 0$, $t > 0$. We shall give, without proof, an expansion which converges in this region and thus has to exhibit explicitly both the poles in $\alpha(s)$ and in $\alpha(t)$. This is

$$
V(s,t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\Gamma(n+1-\alpha_s-\alpha_t)}{\Gamma(n) \Gamma(1-\alpha_s-\alpha_t)} \left( \frac{1}{n-\alpha_s} + \frac{1}{n-\alpha_t} \right)
$$

$$
= -[\alpha(s) + \alpha(t) - 1] \left( \frac{1}{1-\alpha(s)} + \frac{1}{1-\alpha(t)} \right) -
$$

$$
- [\alpha(s) + \alpha(t) - 1] [\alpha(s) + \alpha(t) - 2] \left( \frac{1}{2-\alpha(s)} + \frac{1}{2-\alpha(t)} \right) +
$$

$$
+ ... .
$$

(4.3)

By inspection of the behaviour, as $n \to \infty$, of the terms of the series, Eq. (4.3) is seen to converge for $\text{Re} \ (1-\alpha_s-\alpha_t) < 0$, i.e., for $s + t > (1 - 2\alpha_0)/\alpha' \approx 0$ (Fig. 11). The residues of the poles in Eq. (4.3) grow when $s$ and $t$ increase so that the constructive interference gets stronger and stronger when $s$ and $t$ increase. This is a specific prediction of the present model.

Within the Dalitz plot of Fig. 19 two or three terms of the expansion (4.3) do as well as the complete function. This is also illustrated by Fig. 14 in which three first $s$-channel poles approximate $V(s,t)$ very well. This emphasizes an important point: within a limited kinematical range a dual model can always be approximated arbitrarily well by a non-dual model. However, it is meaningless to truncate the expansion (4.3) and use this form in the $s$-channel physical region for $\pi^+\pi^- + \pi^+\pi^-.$

From a theoretical point of view the justification for Eq. (4.1) is rather slight. The initial state is a $0^-$ state of mass $2M_N$ and it is thus a very low-lying object on the Chew-Frautschi plot, perhaps a particle on the third daughter trajectory of the pion:
The matrix element should, in principle, be one coupling this daughter to three pions, but very little is known about $\alpha_\pi(t)$ and nothing about its daughters. Alternatively, one could say\cite{12} that what is really needed is a fully dual amplitude (with spin and isospin taken into account) for the diagram

\begin{center}
\begin{tikzpicture}
\draw[thick,->] (0,0) -- (2,0);
\draw[thick,->] (0,0) -- (0,2);
\draw[thick,->] (1,0) -- (1,1);
\draw[thick,->] (0,1) -- (1,0);
\draw[thick,->] (0,1) -- (1,1);
\draw[thick,->] (0,1) -- (0,0);
\draw[thick,->] (0,0) -- (0,1);
\draw[thick,->] (1,1) -- (0,0);
\draw[thick,->] (1,1) -- (1,0);
\draw[thick,->] (1,1) -- (0,1);
\node at (0,0) [below left] {$n$};
\node at (1,0) [below right] {$n$};
\node at (0,1) [above left] {$\pi^+$};
\node at (1,1) [above right] {$\pi^-$};
\node at (0.5,0.5) [below] {$\pi^+$};
\node at (0.5,1.5) [above] {$\pi^-$};
\node at (1.5,0.5) [above] {$\pi^+$};
\node at (1.5,1.5) [below] {$\pi^-$};
\node at (0.5,0) [left] {$p$};
\node at (1.5,0) [right] {$p$};
\node at (0.5,1) [right] {$\bar{\pi}^-$};
\node at (1.5,1) [left] {$\bar{\pi}^+$};
\node at (0.5,0.5) [left] {$\bar{\pi}^-$};
\node at (1.5,0.5) [right] {$\bar{\pi}^+$};
\node at (0.5,1.5) [right] {$\bar{\pi}^+$};
\node at (1.5,1.5) [left] {$\bar{\pi}^-$};
\end{tikzpicture}
\end{center}

which, in addition to $\bar{p}n + \pi^+ \pi^- \pi^-$ at rest, describes the same process in flight [i.e., for $s_{\bar{p}n} > (2M_N)^2$], $\pi^+ p \rightarrow \pi^+ \pi^+ n$, $\pi^- p \rightarrow \pi^- \pi^- n$, $\pi^+ n \rightarrow \pi^- \pi^+ p$, etc., by crossing. This should reduce to Eq. (4.1) \cite{12} or to an improvement of Eq. (4.1) when $s_{\bar{p}n} = (2M_N)^2$. Some progress has, in fact, been made in this direction (next Section).
5. THREE-PARTICLE FINAL STATES

Let us first have a look at some experimental data in order to appreciate the similarities and differences between reactions involving four or five particles. We shall use as a model the reaction $K^+ p \rightarrow K^0 \pi^+ p$, which is reasonably easy both to measure and to analyse theoretically (no $K^0 p$ resonances, no Pomeron at present energies, resonance production in $K^0 \pi^+$ and $\pi^+ p$ strongly favoured by isospin).

The presentation of data for two-particle final states is fairly straightforward, since each measured quantity can be a function of two kinematical variables, essentially the total energy and the scattering angle (and perhaps a function of some spin and isospin variables). Thus a three-dimensional space is needed (distribution versus energy and angle) of which one normally presents two-dimensional intersections (e.g., differential cross-section at a fixed energy versus angle). A three-particle final state depends on five variables, since in the c.m. system there are three three-vectors in the final state constrained by four equations expressing four-momentum conservation. Including the total energy this leaves six variables, but one of these is trivial, the orientation around the incoming beam. Thus when presenting data for three-particle final states one always has to make a choice and sum over some variables. This is already imposed by the limited statistics of experimental data.

In connection with dual models the only reasonable alternative is to choose as the five invariant variables $s_{ab} = (p_a + p_b)^2$, $t_{a1} = (p_a - p_1)^2$, $s_{12}$, $s_{23}$, and $t_{b3}$, where the kinematics are defined as follows:

![Diagram](image)

Orienting the momenta as in the reaction $p_a + p_b \rightarrow p_1 + p_2 + p_3$, means that the $s_{ij} \geq (m_i + m_j)^2$ are related to energies and the $t_{ij} \leq (m_i - m_j)^2$ to momentum transfers. However, exactly as in the four-point case, one
may cross to other channels in which the $t_{ij}$ may be of the energy type. Thus, for the reaction $p_0 + p_1 \to p_2 + p_3$, $t_{b1}$ is positive (total energy squared) and $s_{ab}$ negative (one of the momentum transfers). We shall see more examples of the crossing later.

Figure 21 shows the Dalitz plot for $K^+ p \to K^0 \pi^+ p$ and the projections on $s_{K^0 \pi^+}$ and $s_{\pi^+ p}$. The projections are very similar to the two-body cross-section curves shown earlier. At low invariant mass one sees strong resonance formation (in the $K^0 p$ spectrum there are, of course, no resonances) and at higher invariant mass the distribution is smooth [note that a projection in $s_{12}$ ends at $s_{12} = (\sqrt{s} - m_3)^2$, $s$ is the total energy squared]. If one looks at the reaction cross-section $\sigma(K^+ p \to K^0 \pi^+ p)$, one sees that this also decreases smoothly at high energies ($\propto p_{lab}^2$); here there are no resonances in the $s$-channel ($K^+$), but, for example, the three-particle reaction $K^- p \to K^0 \pi^- p$ would contain them. Thus the dependence of the scattering amplitude on an invariant mass does not depend on whether this mass is in the initial state or in the final state, the phase space constraints

---

![Dalitz plot](image)

**Fig. 21** a) The Dalitz plot for $K^+ p \to K^0 \pi^+ p$ at 8.25 GeV/c.
Fig. 21  b) The $K^0\pi^+$ mass spectrum for $K^+p\rightarrow K^0\pi^+p$ at 8.25 GeV/c.

Fig. 21  c) The $\pi^+p$ mass spectrum for $K^+p\rightarrow K^0\pi^+p$ at 8.25 GeV/c.
are just different. This is clearly equivalent to crossing symmetry: the
amplitude describing $K^+ p \rightarrow K^0 \pi^+ p$ will also describe $\pi^+ p \rightarrow K^+ K^0 p$,
$K^0 p \rightarrow K^+ \pi^0 p$, $p \rightarrow K^+ K^0 \pi^-$, etc., and the location of poles is quite indi-

dependent of kinematics. Some poles, however, may be below threshold in
some channels, for example, although the $\Delta(1236)$ pole is seen in $K^+ p \rightarrow K^0 \pi^+ p$
it cannot be seen in $\pi^+ p \rightarrow K^+ K^0 p$ since $2M_K + M_N \approx 1.93 \text{ GeV}$.

Note the difference between the Dalitz plots in Figs. 19 and 21.
In Fig. 19 the interference pattern gets the stronger the higher the spins
of the interfering resonances are, while in Fig. 21 only the first reso-
nances appear prominently. This difference is due to the fact that in
Fig. 19 the $\bar{p}$ annihilates at rest. At higher $p$ momenta one would expect
the Dalitz plot to be more similar to that in Fig. 21.

As the tools for enforcing crossing symmetry have been lacking, the
theoretical analysis done so far has been based on the fact that, apart
from $\approx 10\%$ of "non-resonant background", $K^+ p \rightarrow K^0 \pi^+ p$ goes through the
channels

$$
K^+ p \rightarrow K^{*-}(890) p \\
\quad \rightarrow K^{**}(1420) p \\
\quad \rightarrow K^0 \Delta^{++}(1236). \quad (5.1)
$$

The separation in these channels is not unique, since, for example, there
are events lying simultaneously in the $K^*$ and $\Delta^{++}$ bands. Nevertheless,
these two-body reactions have been investigated theoretically in great
detail and many models, mainly of the exchange type, have been proposed.
For instance, one may try to describe the reaction $K^+ p \rightarrow K^0 \Delta^{++}$ by a model
containing the exchange of the $\rho$ and $A_2$ Regge poles (the exchanged object
has to have $I = 1$, and the $\pi$ does not couple to $K\bar{K}$ since $0^- \neq 0^- 0^-$). The
quantities to be predicted are the differential cross-section $d\sigma/dt_{KK}$,
the decay properties of the $\Delta^{++}$, and the dependence on the total energy
of all these. Reasonably good results are obtained, which indicates that
the basic assumption that parts of the whole reaction behave as two-body
reactions (5.1) is justified. In all these models, the width of the
resonances is neglected and they are treated as stable particles with fixed
$J^P$. As high $J^P$'s are involved, the spin complications are significant
and simplifying assumptions are needed.
In spite of the many successes, the separation in the two-body channels (5.1) is basically unsatisfactory and one would like to have a model which would make it possible to handle all the data at the same time and in a crossing symmetric way. Some progress has been made in this direction. The procedure used is to construct first a basic model function \( B_5(x,y,u,v,w) \) for a process \( p_a + p_b \rightarrow p_1 + p_2 + p_3 \) with only scalar particles involved. This function is analogous to \( B_4(x,y) \) in the two-body case. Next, one tries to construct a model amplitude for a physical reaction, for example, \( KK \rightarrow \pi\pi\pi \), or \( KN \rightarrow \not{K}N \), exactly as

\[
V(s,t) = [1 - \alpha(s) - \alpha(t)]B_4[1 - \alpha(s), 1 - \alpha(t)]
\]

was a reasonable model amplitude for \( \pi\pi \rightarrow \pi\pi \). This can be carried out satisfactorily for some cases involving only mesons, but there is no general recipe for constructing a physical model amplitude for a given reaction.

In the two-body case the basic model function was \( B_4(x,y) \) with \( x = -\alpha(s) \) and \( y = -\alpha(t) \). We are now looking for a function \(^3,^{14},^{15}\) \( B_5(x,y,z,u,v) \) with \( x = -\alpha(s_{ab}), y = -\alpha(t_{a_1}), z = -\alpha(s_{12}), u = -\alpha(s_{23}), v = -\alpha(t_{b_3}) \):

![Diagram](image.png)

and with properties analogous to those of \( B_4(x,y) \).

We shall again first write down the expression for \( B_5 \) and then prove its properties. The form of \( B_5 \) is obtained by generalizing the following integral representation of \( B_4 \):

\[
B_4(x,y) = \int_0^1 du_i u_i^{x-1} (1 - u_i)^{y-1}.
\]

(5.2)
This can be connected with expressions given earlier by expanding \((1 - u_1)^{y-1}\) in powers of \(u_1\):

\[
(1 - u_1)^{y-1} = \sum_{n=0}^{\infty} \binom{y-1}{n} (-u_1)^n
\]

and integrating:

\[
B_s(x,y) = \sum_{n=0}^{\infty} (-1)^n \binom{y-1}{n} \int_0^1 du_1 \, u_1^{x+n-1} \frac{1}{x+n},
\]

which is exactly the expansion given for \(B_s(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)\) in Eq. (2.15). Notice that the integral representation (5.2) converges when \(\text{Re } x > 0\) and \(\text{Re } y > 0\) (when \(\text{Re } x = 0\) the integral diverges at \(u_1 = 0\), when \(\text{Re } y = 0\) at \(u_1 = 1\)) while the series (5.3) converges for all \(x \neq 0, -1, -2, \ldots\) and for \(\text{Re } y > 0\). The series (5.3) involves thus an analytic continuation of Eq. (5.2) in \(x\). If \(x = -\alpha(s), y = -\alpha(t)\), Eq. (5.2) converges for \(s < \alpha, t < \alpha\), i.e., in the \(u\)-channel, while Eq. (5.3) converges also in the \(s\)-channel.

\(B_s\) is then defined by

\[
B_s(x,y,z,u,v) = \int_0^1 \int_0^1 du_1 \, du_u \, u_1^{x-1} \left(\frac{1 - u_1}{1 - u_1 u_u}\right)^{y-1} \left(\frac{1 - u_u}{1 - u_1 u_u}\right)^{z-1} u_u^{u-1} (1 - u_1 u_u)^{v-2},
\]

(5.4)

\[
= \int_0^1 \int_0^1 du_1 \, du_u \, u_1^{x-1} (1 - u_1)^{y-1} (1 - u_1 u_u)^{v-y-z} u_u^{u-1} (1 - u_u)^{z-1},
\]

(5.5)

which converges when the real parts of all the arguments are \(> 0\). The first of these trivially related forms is convenient for proving the cyclic symmetry and for generalizing to still higher \(n\); the second shows how \(B_s\) would be \(B_s(x,y) \cdot B_s(u,z)\) but for the non-factorizable \((1 - u_1 u_u)\) term.

Some properties of \(B_s\) are proved in what follows:

a) **Cyclic and anticyclic symmetry**

\(B_s\) is a model amplitude for a process with identical scalar particles. All channels are thus equal and \(B_s\) should be invariant under all cyclic
and anticyclic permutations of its arguments; i.e., in Fig. 22 it should be irrelevant from which two-particle channel one starts and in which sense one goes around the diagram. This property is a generalization of $B_n(x,y) = B_n(y,x)$. Notice that no amplitudes for realistic reactions would be cyclically symmetric: all channels are not the same.

In order to prove that $B_5(x,y,z,u,v) = B_5(y,z,u,v,x)$, introduce in Eq. (5.4) the new variables

$$v_i = \frac{1 - u_i}{1 - u_i u_n},$$

$$v_u = 1 - u_i u_n,$$  \hspace{1cm} (5.6)

or conversely

$$u_i = 1 - v_1 v_u,$$

$$u_u = \frac{1 - v_u}{1 - v_1 v_u}. \hspace{1cm} (5.7)$$

Calculating the Jacobian of the transformation one finds

$$du_1 du_u = \frac{\partial(u_i, u_u)}{\partial(v_i, v_u)} dv_1 dv_u = \frac{v_u}{1 - v_1 v_u} dv_1 dv_u. \hspace{1cm} (5.8)$$

The form of the Jacobian explains why one has the power $v - 2$ in Eq. (5.4); one $1 - u_i u_n$ is cancelled by the $v_u$ in the Jacobian. Inserting Eqs. (5.7) and (5.8) in Eq. (5.4) one finds immediately the desired result

$B_5(x,y,z,u,v) = B_5(y,z,u,v,x)$. The anticyclic symmetry in the form

$B_5(x,y,z,u,v) = B_5(u,z,y,x,v)$ is finally obtained from Eq. (5.5) by just relabelling $u_i \leftrightarrow u_u$.

b) Pole structure

The poles of $B_5$ arise again due to a divergence of the integral representation (5.5) or its analytic continuations. Let us consider the poles in $u$ by writing Eq. (5.5) in the form

$$B_5(x,y,z,u,v) = \int_0^1 du_u u_u^{u-1} f(u_u), \hspace{1cm} (5.9)$$
where
\[
f(u_n) = \int_0^1 du_1 \, u_1^{x-1}(1-u_1)^{y-1} \left(1 - u_1 u_n\right)^{y-z-y(1-u_n)^{z-1}}.
\] (5.10)

If \( f(u_n) \) is expanded in powers of \( u_n \):
\[
f(u_n) = \sum_{n=0}^{\infty} \frac{1}{n!} \, f^{(n)}(0) \, u_n^n
\] (5.11)

and inserted in Eq. (5.9), one obtains
\[
B_5(x,y,z,u,v) = \sum_{n=0}^{\infty} \frac{R_n(x,y; z,v)}{u + n},
\] (5.12)

where by expanding the two last terms in Eq. (5.10) as a power series in \( u_n \), taking the derivative, and setting \( u_n = 0 \):
\[
R_n(x,y; z,v) = \frac{1}{n!} \, f^{(n)}(0)
\] (5.13)

\[
= (-)^n \sum_{k=0}^{n} \binom{v-2-y}{k} \binom{z-1}{n-k} B_n(x + k, y).
\]

In particular, the two first terms of Eq. (5.12) are
\[
B_5(x,y,z,u,v) = B_n(x,y) \cdot \frac{1}{u} + B_n(x,y) \left[ 1 + \frac{xy - yz - xv}{x + y} \right] \frac{1}{1 + u} + \ldots.
\]

Pictorially we have now represented the amplitude for \( p_a + p_b + p_1 + p_2 + p_3 \) as a sum over resonances in the 23 channel or, by cyclic symmetry, in any channel:
The expansion (5.12) continues the integral representation analytically to \( u < 0 \); it converges when \( x, y, z, v \) are all positive. Physically, if \( x = -\alpha(s_{ab}) \), etc., this corresponds to a channel in which \( s_{ab} < 0, t_{a_1} < 0, s_{12} < 0, t_{b_3} < 0 \). This is the channel in which \( p_b \) and \( p_1 \) are incoming.

Consider then the pole at \( u = 0 \). The amplitude has been factorized into three parts, \( B_u(x,y) \) describing the quasi two-body reaction producing the particle of mass defined by \( \alpha(s_{23}) = \alpha_0 + \alpha's_{23} = 0 \), the propagator \( 1/u \) and the decay of the particle. There is no dependence on the variables \( z \) and \( v \) connecting the production and decay and thus the intermediate particle has spin zero, as it should for \( \alpha(s_{23}) = 0 \). On the pole at \( u = -1 \) or \( \alpha(s_{23}) = 1 \) there is an additional factor which depends linearly on \( z \) and \( v \). This corresponds to spin 1 and a spin 0 daughter. In general, at \( u = -\ell \) the residue is a polynomial of order \( \ell \) in \( z \) and \( v \). The spin analysis is now rather complicated, since the intermediate spin \( \ell \) particle decays into three particles.

c) Double Regge behaviour

When generalizing the \( B_u \) result (\( y \) fixed, \( x \to \infty \))

\[
B_u(x,y) = \frac{\Gamma(y)}{\Gamma(x+y)} \left[ \int_0^\infty dt_1 t_1^{\ell-1} e^{-t_1} \right] x^y
\]

(5.14)

for \( B_5 \), one has to be careful in defining how one goes to \( \infty \) since there are many different possible ways. First one has to avoid the poles on the negative real axes, but this is done as with \( B_u \). Forgetting this difficulty we shall consider the limit in which the two subenergies \( s_{12} \) and \( s_{23} \) simultaneously increase above the resonance region so that one may approximate
To obtain this double Regge limit, one clearly has to let \( x \to \infty (s_{ab} \to \infty) \) for constant \( y \) and \( v \) (constant momentum transfers \( t_{a1} \) and \( t_{b3} \)). In addition, one must have \( z \to \infty \) and \( u \to \infty (s_{12} \to \infty \) and \( s_{23} \to \infty) \), but it will appear that a meaningful limit is only obtained if \( \lambda = zu/x \) (or \( s_{12}s_{23}/s \)) remains constant. Here this is just a mathematical fact (\( B_5 \) knows nothing about phase space), but it so happens that from pure kinematics or phase space one can prove the following related inequality for large \( s_{12} \) and \( s_{23} \):

\[
m_2^2 + (\sqrt{-t_{a1}} + \sqrt{-t_{b3}})^2 \geq \frac{s_{12} s_{23}}{s}.
\]

This shows that if one keeps \( t_{a1} \) and \( t_{b3} \) constant, \( s_{12}s_{23}/s \) has to remain bounded. Thus the mathematical limit defined above as the double Regge limit is important since it lies within the physical region; there may be other non-physical limits. As the result for the limit is very suggestive, we give it here without derivation:

\[
B_5(x,y,z,u,v) \to \left[ \int_0^{\infty} dt_1 dt_2 t_1^{y-1} t_2^{v-1} e^{-t_1 - t_2 - t_1 t_2/\lambda} \right] z^{-y} u^{-v}.
\]

Apart from the factor \( \exp (-t_1 t_2/\lambda) \) this is formula (5.14) taken twice. The factor in square brackets in formula (5.15) is an example of a Reggeon-Reggeon-particle residue function.

We have now at our disposal a good zero-width dual model function for a world with only scalar external particles and resonances lying on trajectories \( \alpha(s) = \alpha_0 + \alpha's \) with \( \alpha_0 < 0 \). In order to illustrate the problem of constructing physical model functions, we shall choose a case which on the five-point level is simple in the same sense as \( \pi\pi \) scattering was simple in the four-point case. This is one with only \( 0^- \) external mesons and with resonances lying only on the three trajectories of \( 1^- \) and \( 2^+ \) mesons: \( \alpha_\rho, \alpha_K^* \) and \( \alpha_\phi \) [Eq. (3.5)]. Due to G-parity conservation one cannot work with only five pions and we shall instead consider \( K\bar{K} \to K\pi K \) in the charge state \( K^-K^+ \to K^-\pi^+K^0 \), keeping in mind the transition to \( \bar{K}N \to \bar{K}N \).

This charge state was chosen so as to have as many exotic channels as possible. In \( \pi\pi \) scattering \( \pi^+\pi^- \to \pi^+\pi^- \) had one exotic channel (u-channel) and out of three possible terms \( V(s,t), V(t,u) \) and \( V(s,u) \),
only one, \( V(s,t) \) remained. We can represent these three terms as follows:

![Graph with nodes and edges]

\[\begin{array}{c}
A & 1 & 2 \\
B & 3 & 1 \\
A & 1 & 2 \\
B & 3 & 1 \\
A & 2 & 1 \\
B & 3 & 1 \\
A & 3 & 1 \\
B & 2 & 1 \\
A & 2 & 1 \\
B & 3 & 1 \\
A & 2 & 1 \\
B & 1 & 1 \\
\end{array}\]

Fig. 23 The 12 possible diagrams not connected by cyclic or anticyclic permutations of the five particles involved. All particles are taken to be incoming.
On the five-point level one has similarly 12 different terms, given in Fig. 23. The given 12 orderings of a, b, 1, 2, 3 are those of a total of $5! = 120$ permutations which cannot be connected by cyclic or anticyclic permutations. Now it is also very convenient to consider all particles as incoming, i.e., for any outgoing particles change the direction of the line and the particle into antiparticle. Then it is easy to see which channels are exotic.

For $K^- K^+ \to K^- \pi^+ K^0$ the first diagram of Fig. 23 has the form

![Diagram](attachment:image.png)

One sees that the channels $K^+ K^+$ (b1), $K^- \pi^-$ (a2), $K^- K^0$ (a3) are exotic and thus any of the diagrams in which these channels appear do not contribute. In fact, only the diagram 2, which differs from diagram 1 by a permutation of 2 and 3 does not contain these channels. Thus the existence of three exotic channels reduced the number of diagrams to two. Two exotic channels give similarly four diagrams and one exotic channel six diagrams.

In order to see which trajectories couple to which channels we shall, following the example of the previously analysed case of $KK \to KK$, use duality diagrams. Connected duality diagrams can be drawn as previously in the four-point case, and if any channels are exotic one knows that the corresponding duality diagram will be non-planar. In addition, it may happen that even if there are no exotic channels, a planar duality diagram cannot be drawn (as in $K^- p \to \pi^- \pi^+$). In $K^- K^+ \to K^- \pi^+ K^0$ a planar duality diagram can be drawn for the two diagrams 1 and 2 of Fig. 23:
The duality diagrams show here again their great usefulness: without any further calculations one sees that the following trajectories couple in the diagrams 1 and 2 of Fig. 23:

where \( \rho, K^*, \phi \) denote the three trajectories of Eq. (3.5).

In order to construct the amplitudes we shall again proceed by associating a model function with each duality diagram. Since all the trajectories have their first particles at \( \alpha_i(s) = 1, i = \rho, K^*, \phi \), one has to use, as the arguments of \( B_5, 1 - \alpha_i(s) \). The contributions \( D_1 \) and \( D_2 \) of the two diagrams are then

\[
D_1 = K \cdot B_5(1) \equiv K \cdot B_5[1 - \alpha_\phi(s_{ab}), 1 - \alpha_\rho(t_{a1}), 1 - \alpha_K(s_{12}), 1 - \alpha_K(s_{23}), 1 - \alpha_\rho(t_{b3})]
\]

\[
D_2 = K \cdot B_5(2) \equiv K \cdot B_5[1 - \alpha_\rho(s_{ab}), 1 - \alpha_\phi(t_{a1}), 1 - \alpha_\rho(s_{13}), 1 - \alpha_K(s_{23}), 1 - \alpha_\rho(t_{b2})]
\]

(5.16)

where \( K \) is a kinematical factor, analogous to \( 1 - \alpha(s) - \alpha(t) \) in \( \pi\pi + \pi\pi \).

The new kinematic variables \( s_{13} = (p_1 + p_3)^2 \) and \( t_{b2} = (p_b - p_2)^2 \) in \( D_2 \) are simply related to those in \( D_1 \) as follows:

\[
s_{13} = s - s_{12} - s_{23} + m_1^2 + m_2^2 + m_3^2
\]

\[
t_{b2} = t_{a1} - t_{b3} - s_{23} + m_b^2 + m_3^2 + m_2^2.
\]

(5.17)
The total amplitude is $D_1 + D_2$, since no $\pi^0$ or $\eta$ is involved. One can also see this by crossing to the channel $K^-\bar{K}^0 \to K^-K^+\pi^0$; then the amplitude clearly has to be symmetric under the exchange of the $K^-$ momenta, which imposes the combination $D_1 + D_2$.

$K$ is needed in Eqs. (5.16) since although the poles of Eqs. (5.16) are in the correct positions $\alpha(s) = 1, 2, \ldots$, the residue of the pole at $\alpha(s) = \lambda$ is only a polynomial of order $\lambda - 1$ in the variables connecting the two parts into which the pole $\alpha(s) = \lambda$ divides the five-point function. Connected with this is the fact that the double Regge behaviour is not what is to be expected; if $s_{ab} \to \infty$ then by Eq. (5.15)

$$D_1 \to K \times \text{residue function} \times (-\alpha'_s s_{12})^{\alpha_p(t_{a1})^{-1}} (-\alpha'_s s_{23})^{\alpha_p(t_{b3})^{-1}}.$$ (5.18)

$K$ thus has to behave in this limit as $K \to \text{constant} \cdot s_{12}s_{23} = \text{constant} \cdot \lambda s$, since $\lambda = s_{12}s_{23}/s$ is kept constant.

The solution for $K$ is suggested by different arguments, for example, by Feynman rules with intermediate vector particles. The essentially unique result is

$$K = \varepsilon_{\mu\nu\rho\sigma} p_\mu^a p_\nu^b p_\rho^1 p_\sigma^2 \equiv \varepsilon[p_a, p_b, p_1, p_2] \cdot$$ (5.19)

As this kinematical factor has been and will be widely used, we shall, in the following, list some of its properties in a fairly detailed manner:

i) Since the complete amplitude has to be scalar and since the five external particles are pseudoscalar $K$ is pseudoscalar. That $K \to -K$ under the parity transformation is evident from Eq. (5.20).

ii) The $\varepsilon$-tensor is the totally antisymmetric fourth-rank tensor with $\varepsilon_{0123} = 1$ and thus the definition of $K$ is, in fact, a definition of a determinant in the components of the $p_i$ ($p_i^a = E_a$, etc.):

$$K = \begin{vmatrix} p_1^0 & p_2^0 & p_1^0 & p_2^0 \\ p_1^1 & p_2^1 & p_1^1 & p_2^1 \\ p_1^2 & p_2^2 & p_1^2 & p_2^2 \\ p_1^3 & p_2^3 & p_1^3 & p_2^3 \end{vmatrix}.$$ (5.20)

The components may be defined in an arbitrary frame.
iii) Since the \( \epsilon \)-tensor is totally antisymmetric, one has, for example,
\[
\epsilon [p_a, p_b, p_1, p_2] = 0 \quad \text{and since Eq. (5.19) is linear in its arguments}
\]
\[
K = \epsilon [p_a + p_b, p_1, p_2] = \epsilon [p_a, p_b, p_1, p_1 + p_2],
\]
eq etc. Also any of the \( p_1 \) in Eq. (5.19) may be replaced by \( p_3 \) by using four-momentum conservation \((p_a + p_b = p_1 + p_2 + p_3)\).

iv) Take \( K \) from Eq. (5.20), multiply the first row and column by \(-1\)
(all signs changed except that of \( p^0_a \)), transpose, and finally multiply by \( K \) as defined in Eq. (5.20). The first element of the result is \((p^0_a)^2 - p^2_a = p^2_a\), and continuing one obtains
\[
K^2 = -\Delta_4 = \begin{vmatrix}
    p_a^2 & p_a \cdot p_b & p_a \cdot p_1 & p_a \cdot p_2 \\
    p_b^2 & p_b \cdot p_1 & p_b \cdot p_2 & \\
    p_1^2 & p_1 \cdot p_2 & & \\
    p_2^2 & & & 
\end{vmatrix}.
\]

The symmetric determinant \( \Delta_4 \) is the Gram determinant of the vectors \( p_a, p_b, p_1 \) and \( p_2 \). The scalar products in Eq. (5.22) are easily expressed in terms of the invariants by \( s_{ij} = m_i^2 + m_j^2 + 2p_i \cdot p_j \)
etc. It is then easy by neglecting the masses to find out that when \( s_{ab} \rightarrow \infty \)
\[
\Delta_4 \rightarrow \text{constant} \times s_{12}^2 s_{23}^2,
\]
where the constant depends on the constant values of \( t_{a_1}, t_{b_3} \) and \( s_{12}s_{23}/s_{ab} \). According to Eq. (5.18) the correct double Regge behaviour is then obtained. Since \( K \) is symmetric and since the \( B_s \)'s in Eq. (5.16) contain \( 1 - \alpha_l(s) \) in all the arguments this holds for all possible channels.

v) \( K \) can be put in a very simple form if one goes to the rest frame of any of the four-vectors involved. For instance, in the total c.m. frame, one has \( p_a + p_b = (\sqrt{s_{ab}}, 0) \). Thus, in Eq. (5.21) only the time component of \( p_a + p_b \) contributes and
\[
K = \sqrt{s_{ab}} \epsilon_{0ijk} p_b^i p_1^j p_2^k = \sqrt{s_{ab}} p_b \cdot (p_1 \times p_2),
\]
where the three-vectors are defined in the total c.m. frame. We shall later show how Eq. (5.24) implies the correct spin structure at the poles.

vi) From the above one can see that $K$ vanishes in degenerate situations, for example, if some of the momentum vectors are parallel or all lie in the same plane. These situations correspond to the boundaries of the physical region. In fact, one can prove that within the physical region always $K^2 = -\Delta_4 > 0$ and that at the boundary $\Delta_4 = 0$.

Our model amplitude $D_1 + D_2$ for $K^-K^+ \to K^-\pi^+K^0$ is now completely fixed (apart from a normalization factor). For any comparison with (hypothetical) data one again has to smooth the poles, for example, by Eq. (2.21). If the amplitude $^6$ then is fed into a Monte Carlo program one may calculate any distribution one might want. Qualitatively it is obvious that $D_1$ will produce on the $s_{12}$ versus $s_{23}$ Dalitz plot strong $K^*$, $K^{**}$, etc., bands in both variables so that the importance of the resonance decreases with increasing spin. $D_2$ will then produce $K^*$ effects in $s_{23}$ and an $A_2$ band in $s_{13}$ (remember that the $\rho$ is below the $K\bar{K}$ threshold). There will be interference between $D_1$ and $D_2$, but this is now completely determined.

In addition, and this is very important, one may use $D_1 + D_2$ without any change to calculate predictions for any channel connected to $K^-K^+ \to K^-\pi^+K^0$ by crossing. Thus, if one goes to the channel in which $t_{b1}$ is the total energy one has the reaction $K^+K^+ \to K^+\pi^-K^0$. This corresponds to the crossing (line reversal) from $p_a + p_b + p_1 + p_2 + p_3$ to $\bar{p}_1 + \bar{p}_b + \bar{p}_a + p_2 + p_3$. For clarity, it may be convenient to relabel the particles (by interchanging $p_a$ and $p_1$), so that $K^+K^+ \to K^+\pi^-K^0$ is now $p_a + p_b + p_1 + p_2 + p_3$. The amplitude $D_1$ for $K^+K^+ \to K^+\pi^-K^0$ is then, calling energy (momentum transfer) type variables $s_{ij}(t_{ij})$:

$$D_1 = K \cdot B_{kl} \left[ 1 - \alpha(t_{b1}), 1 - \alpha(t_{a1}), 1 - \alpha(t_{a2}), 1 - \alpha(t_{a3}), 1 - \alpha(t_{b3}) \right]$$

(5.25)

and a similar expression is valid for $D_2$. Now only one argument, $1 - \alpha(t_{s23})$, gives rise to poles, since all the others contain momentum transfers. This, of course, is just the input, since in the final state only the com-
combination $\pi^+k^0$ is non-exotic. On the four-point level Eq. (5.25) corresponds to a $(u,t)$ term.

Similarly one may cross to the channel in which $s_{23}$ is the total energy. The corresponding reaction is $\pi^+k^0 \to K^-K^+K^*$. Now $D_1$ will, on the Dalitz plot, give rise to a $\phi$ and recurrences in one of the $K^-K^+$ combinations and an $A_2$ and recurrences in the other one. $D_2$ will have the roles of the $K^-K^+$ combinations inverted, to maintain the symmetry.

Let us then look in more detail at the production of the first $K^*$ pole, the $K^*(890)$. We can find the amplitude immediately by going in Eqs. (5.16) to the pole at $\alpha_{K^*}^0(s_{23}) = 1$ and by taking the residue from Eq. (5.13):

$$A(K^-K^+ + K^-K^* + K^-K^0) = \varepsilon[p_b, p_1, p_2, p_3] \times \frac{1}{1 - \alpha_{K^*}^0(s_{23})} A(s_{ab}, \tau_{ai})$$

(5.26)

where

$$A(s_{ab}, \tau_{ai}) = B_a[1 - \alpha_\phi(s_{ab}), 1 - \alpha_\psi(t_{ai})] + B_a[1 - \alpha_\phi(s_{ab}), 1 - \alpha_\psi(t_{ai})]$$

(5.27)

For $A(K^+K^- + K^+K^*)$ one would obtain from Eq. (5.25) just the same result except for a replacement of $s_{ab}$ by $t_{b1}$, i.e., of $s$ by $u$. The two terms in Eq. (5.27) correspond evidently to the two terms of $A(K^-K^+ + K^-K^*)$ in Eq. (3.6).

The kinematic factor in Eq. (5.26) has been written in a form which makes it easy to evaluate in the rest frame of the produced $K^*$, $p_{23} = p_2 + p_3 = 0$. This frame is very convenient to use if one wants to analyse the decay properties of a resonance. Let us choose in this frame a Cartesian coordinate system (Fig. 24) so that $p_b(=p_1 - p_a)$ defines the $z$-axis and so that the reaction plane (the plane containing $p_a + p_b + p_1 + p_{23}$, $p_{23}$ now being zero) coincides with the $xz$ plane. If the polar and azimuthal angles of $p_3(= -p_2)$ are $\theta_{b3}$ and $\phi$ (these are the Jackson and Treiman-Yang angles, respectively), the $\varepsilon$-factor can be written as:

$$\varepsilon[p_b, p_1, p_2, p_3] = \sqrt{s_{23}} \frac{p_b \cdot (p_1 \times p_3)}{|p_3|}$$

$$= |p_2| \times |p_3| \sqrt{s_{23}} \sin \theta_{b3} \sin \phi$$

(5.28)
Fig. 24  The definition of the polar ($\theta_{b3}$) and azimuthal ($\phi$) angles in the frame $p_2 + p_3 = 0$, or $p_b = p_1 - p_a$.

Here $p_b \times p_1$ is perpendicular to the scattering plane and it is thus invariant under all Lorentz transformations in the scattering plane. In particular, it is invariant if we transform to the reaction c.m. frame, $p_a + p_b = 0$:

$$|p_b \times p_1| = |p_b^{\text{c.m.}} \times p_1^{\text{c.m.}}| = p_1 p_f \sin \theta,$$

in an evident notation. We have thus succeeded in splitting $A(K^- K^+ + K^- K^* + K^- \pi^+ K^0)$ in three parts: an amplitude

$$A(K^- K^+ + K^- K^{*+}) = p_1 p_f \sin \theta \cdot A(s_{ab}, t_{ai})$$  \hspace{1cm} (5.29)

describing the quasi two-body process $K^- K^+ + K^- K^{*+}$, the propagator $1/[1 - \alpha_{K^*}(s_{23})]$, and an amplitude describing the decay of the $K^*$ in its rest frame:
\( \Lambda(K^{*+} \to K^0 \pi^+) = \sqrt{s_{23}} |p_3| \sin \theta_{b3} \sin \phi \). \hspace{1cm} (5.30)

Note that the momentum vectors can be easily written in terms of invariants:

\[ p_i = \frac{1}{4} \sqrt{s_{ab} - 4m_K^2}, \quad p_f = \sqrt{\lambda(s_{ab}, m_{K^*}^2, m_{K^*}^2)/2s_{ab}}, \]

\[ \sqrt{s_{23}} |p_3| = \frac{1}{2} \sqrt{\lambda(s_{23}, m_K^2, m_{\pi}^2)}. \]

From Eq. (5.29) one sees that the production amplitude vanishes in the forward and backward directions as \( \sin \theta \). This is characteristic of spin flip. The decay angular distribution in the \( K^* \) rest frame is, according to Eq. (5.30),

\[ w(\cos \theta_{b3}, \phi) = \frac{3}{4\pi} \sin^2 \theta_{b3} \sin^2 \phi. \hspace{1cm} (5.31) \]

It has become customary to parametrize \( w(\cos \theta, \phi) \) in terms of density matrix elements \( \rho_{mm}^{17} \). A comparison with the standard expression

\[ w(\cos \theta, \phi) = \]

\[ \frac{3}{4\pi} \left[ \rho_{11} \sin^2 \theta + (1 - 2\rho_{11}) \cos^2 \theta - \rho_{-1,-1} \sin^2 \theta \cos 2\phi - \sqrt{2} \Re \rho_{10} \sin 2\theta \cos \phi \right] \]

shows that Eq. (5.31) is obtained if \( \rho_{11} = \rho_{1,-1} = \frac{1}{4} \), all other elements being zero. This result has been obtained in the duality framework, but one should emphasize that it follows entirely from the kinematic factor. In fact, already the fact that in \( \bar{K}K \to \bar{K}K^* \) one has only natural parity (as \( 1^+ \) or \( 2^+ \) in the t-channel) predicts that \( \rho_{11} = \frac{1}{4} \) and that of the other elements only \( \rho_{1,-1} \neq 0 \).

The result [Eq. (5.31)] shows also that the spin structure of the model is correct. In fact, at the \( K^* \) position, one has only P-wave [Eq. (5.31) has no constant terms] and the situation thus is simpler than in the \( \pi\pi \) case, where at the \( \rho \) position one had so much S-wave that it even violated unitarity. The reason for this is that only the kinematic factor contributes to Eq. (5.31). The structure of \( B_s \) starts affecting the results if one considers the production of the second pole, the \( 2^+ \)

\( K^{**}(1420) \). Then one should take into account the second term of Eq. (5.13),
which introduces a residue function depending linearly on the momentum transfers. Now the decay distribution will contain both a $2^+$ parent and a $1^-$ daughter contribution, and the decay distribution or the density matrix elements will also depend on $s_{ab}$ and $t_{ai}$.

We have considered here a particular charge state having as many exotic channels as possible. The general treatment of all charge states is again carried out most simply in terms of the isospin amplitudes (there are three of them), but we shall not go into details here.

Equipped with such a beautiful ansatz for a five-point amplitude one would, of course, like to see whether it could be used for data analysis. This has, in fact, been tried for several reactions\(^\text{16}\). We shall here limit ourselves to listing some difficulties one has to face if, in $K^-K^+ \to K^-\pi^+K^0$, one replaces the $K$ isodoublet by the $N$ isodoublet, i.e. considers the reaction $K^-p \to K^-\pi^+n$. This listing could have been done when going from $K^-K^+ \to K^-K^+$ to $K^-p \to K^-p$, and one cannot expect any analysis of $\bar{K}N \to \bar{K}NN$ to work better than $\bar{K}N \to \bar{K}N$.

i) Since the baryon has spin $\frac{1}{2}$ there are four invariant amplitudes (now the parity restriction reducing the number in $\bar{K}N \to \bar{K}N$ to two does not operate). Correspondingly, there is no unique kinematic factor. The baryon spin also introduces the problem of parity doubling, i.e. the model predicts states [like a $\frac{3}{2}^- \Delta(1235)$] not found in nature.

ii) More trajectories couple. For instance, the $\pi$ often couples strongly to $\bar{N}N$. However, as a low-lying unnatural parity trajectory, the $\pi$ does not have simple duality properties and it is unclear whether there is any meaning in putting it in a dual model. Also both the $\Lambda$ and $\Sigma$ trajectories, which definitely are non-degenerate, couple simultaneously to $K^-p$, while in $K^-K^+$ one had in one term either $\rho$ or $\phi$.

One way of getting around these difficulties\(^\text{16}\) is to neglect the baryon spin and to consider a channel in which experimentally vector mesons couple most strongly to $NN$ (these are the channels with a $NN$ system of zero charge: $pp$ or $n\bar{n}$). Then the analogy to $\bar{K}K \to \bar{K}nK$ is considerable: one has trajectories with $P$-wave first resonances in all channels. In particular, the use of the $\epsilon$-factor may then be motivated. With a fairly
credible set of assumptions an amplitude for $\bar{K}N \to \bar{K}nN$ can thus be constructed and predictions for this channel and all channels connected with it by crossing ($KN \to K\pi N$, $\pi N \to K\bar{K}N$, $\bar{N}N \to K\bar{K}N$) can be computed. Taking into account the large amount of data handled the results look very good, but it is hard to say to what extent they form a test of the model and of the details of $B_s$, or to what extent they already essentially follow from the properties of Regge poles and of the kinematical factor. What is undeniable, in any case, is that the $B_s$ is a very compact way of representing an amplitude having, on the average, good properties in all channels.
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