D-string fluid in conifold:
I. Topological gauge model

R. Ahl Laamara$^{1,2}$, L.B Drissi$^{1,2}$, E.H Saidi$^{1,2,3}$

1. Lab/UFR-Physique des Hautes Energies, Faculté des Sciences de Rabat, Morocco.
2. Groupement National de Physique des Hautes Energies, GNPHE;
   Siege focal, Lab/UFR-HEP, Rabat, Morocco.
3. VACBT, Virtual African Centre for Basic Science and Technology,
   Focal point Lab/UFR-PHE, Fac Sciences, Rabat, Morocco.

April 4, 2006

Abstract

Motivated by similarities between quantum Hall systems à la Susskind and aspects of topological string theory on conifold as well as results obtained in hep-th/0601020, we study the dynamics of D-string fluids running in deformed conifold in presence of a strong and constant RR background B-field. We first introduce the basis of D-string system in fluid approximation and then derive the holomorphic non commutative gauge invariant field action describing its dynamics in conifold. This study may be also viewed as embedding Susskind description for Laughlin liquid in type IIB string theory. FQH systems on real manifolds $R \times S^2$ and $S^3$ are shown to be recovered by restricting conifold to its Lagrangian sub-manifolds. Aspects of quantum behaviour of the string fluid are discussed.

Key words: Quantum Hall fluids, D string in conifold, topological gauge theory, non commutative complex geometry.

*h-saidi@fsr.ac.ma
Contents

1 Introduction

2 D fluid Model proposal
   2.1 D string variables
   2.2 Fluid approximation

3 Field action
   3.1 Classical B_{RR}-D string coupling
   3.2 Implementing density constraint equation

4 Holomorphy and quantum corrections
   4.1 Quantum effects and conifold deformations
   4.2 Supersymmetric embedding
      4.2.1 Holomorphy property and boundary QFT
      4.2.2 Supersymmetric interpretation

5 Conclusion and outlook
1 Introduction

Since Susskind proposal on fractional quantum Hall (FQH) fluids in Laughlin state as systems described by $(2 + 1)$ non commutative CS gauge theory \cite{1}, there has been a great interest for building new solutions extending this idea \cite{2}-\cite{6}. Motivated by: (a) results concerning attractor mechanism on flux compactification \cite{7, 8}, in particular the link with non commutative geometry, and (b) the study of \cite{9} dealing with topological non commutative gauge theory on conifold, we develop in this paper a new extension of Susskind proposal for FQH fluids to higher dimensions. Our extension deals with modelization of the dynamics of a fluid of D strings running in conifold and in presence of a strong and constant RR background B-field. The extended system lives in complex three (real six) dimensions and is related to the usual FQH system with point like particles by the following correspondence:

(1) The role of the usual FQH particles moving in a real Riemann surface $\mathcal{M}$ with coordinates $z$ and $\bar{z}$, is played by D strings moving on K3 surface with some complex holomorphic coordinates $u$ and $v$ to be specified later. In this picture, FQH particles may be then viewed as D0 branes coming from D1 strings wrapped on $S^1$.

(2) The complex coordinates $z_a(t)$ and $\bar{z}_a(t)$ parameterizing the dynamics of the $N$ fractional quantum Hall particles are then mapped to $u_a = u_a(\xi)$ and $v_a = v_a(\xi)$ with $\xi = t + i\sigma$ being the string world sheet complex coordinate.

(3) The local coordinates $(t, z, \bar{z})$ parameterize a real three dimension space; say the space $R^{1,2}$. The local variables $(\xi, u, v)$ parameterize a complex three dimension space, which is just the conifold $T^*S^3$ realized as $T^*S^1$ fibered on $T^*S^2$. The $R^{1,2}$ geometry used in Susskind description appears then as a special real three dimension slice of conifold.

(4) The role of the magnetic field $B$ is now played by a constant and strong RR background field $B$ of type IIB string. Like in FQH system, the B field is supposed normal to K3 surface and strong enough so that one can neglect other possible interactions.

From this naive and rapid presentation of the higher dimensional extended FQH system, to which we refer here below as a D-string fluid (DSF for short), one notes some specific properties among which the three following: First, Susskind proposal may be recovered from DSF by taking appropriate parameter limits of DSF moduli space to be described later. Second, the real geometry of FQH system is contained in conifold; the present study may be then thought of as embedding Susskind field theoretical model for Laughlin state with filling factor $\nu = \frac{1}{k}$ into type IIB superstring theory on conifold. This property offers one more argument for embedding FQH systems in supersymmetric theories; others arguments have been discussed in \cite{10, 11}. Finally, in DSF model, the complex holomorphy property plays a basic role; reality is recovered by restricting
conifold to its half dimension Lagrangian sub-manifold. This involution has the effect of projecting DSF into the usual FQH system opening the way for links between real 3D physics and type II superstrings on Calabi-Yau threefolds.

The presentation of this paper is as follows: In section 2, we introduce the basis of fluid approximation of D-strings running in conifold. To build this system, we use special properties of K3 complex surface and conifold geometry. We also take advantage of Susskind model for Laughlin liquid which we use as a reference to make comparisons and physical interpretations. In section 3, we study the classical dynamics of the interaction between D strings and the RR magnetic background field. We suppose that B is strong enough so that one can neglect string kinetic energy and mutual energy interactions between the D strings. We also suppose that the number of D strings per volume unit is high and uniform. Then use the fluid approximation to derive the effective field theory extending Susskind model. In this section, we also study some special limits such as real projection. In section 4, we discuss quantum aspects of the D-strings fluid, in particular holomorphic property and in section 5 we give our conclusion and outlook.

2 D fluid Model proposal

Like in usual fractional quantum Hall fluids in real three dimensions, the D string system we consider here involves, amongst others, two basic ingredients:

(a) A set of N D strings running in conifold and printing a line trajectory $T$ on the complex two surface K3. The curve $T$ is exactly the world line trajectory one gets if the D strings $(u(\xi), v(\xi))$ collapse to point like particles $(z(t), \bar{z}(t))$.

(b) A constant RR background field B, which is taken normal to K3, governs the dynamics of the strings. The magnitude of the B field is supposed strong enough such that one can neglect all other interactions in the same spirit as we do in FQH systems involving point like particles. Non zero B field induces then a non commutative geometry on K3 captured by the Poisson bracket \[ \{X(\xi, u, v), Y(\xi, u, v)\}_{u,v} \sim \partial_u X \partial_v Y - \partial_v X \partial_u Y \] of the dynamical variables $X(\xi, u, v)$ and $Y(\xi, u, v)$ of the fluid approximation.

To get the gauge invariant effective field action $S_{DSF}$ describing the dynamics of fluids of D strings in conifold with analogous conditions as in FQH systems, we need two essential things. First fix the classical field variables $u = u(\xi)$ and $v = v(\xi)$ describing the D string dynamics in conifold and second implement the fluid approximation by using a uniform particle density $\rho = \rho(u, v)$ to deal with the number of D strings per volume unit. We know how this is done in the case of standard FQH fluids in Laughlin state with filling fraction $\nu = \frac{1}{k}$ and we would like to extend this construction for D string fluids taken in similar conditions. Though the geometries involved in the present
study are a little bit complicated and the basic objects are one dimensional extended elements, we will show that the theoretical analysis is quite straightforward.

For the choice of the string variables $u = u(\xi)$ and $v = v(\xi)$; they are given by the geometry of K3 and the fluid description is obtained by extending Susskind analysis for FQH particles. For fluid approximation, we use also properties of holomorphic area preserving diffeomorphisms on K3 demanding a uniform density. Seen that the idea of the general picture has been exposed before and seen that details requires involved tools, we begin the present analysis by describing, in next subsection, the D string dynamical variables. Then come back to the fluid approximation with uniform density. More details on the holomorphic gauge invariant field action and real truncating will be considered in the forthcoming section.

2.1 D string variables

First note that there are various kinds of K3 surfaces; the one we will be using below is a local K3 with a deformed $A_1$ singularity; that is $T^*P^1 \simeq T^*S^2$. Second note also that the complex surface K3 is a non flat Kahler manifold and so the natural way to define it is in term of a projective surface embedded in a homogeneous complex three space as given below,

$$xy - zw = \mu, \quad x, y, z, w \in \mathbb{C}, \quad (2.1)$$

together with the following projective transformations,

$$(x, y, z, w) \rightarrow \left(\lambda x, \frac{1}{\lambda} y, \lambda z, \frac{1}{\lambda} w \right), \quad (2.2)$$

and where $\mu$ is a complex constant. In these relations, we have four complex holomorphic variables namely $x, y, z$ and $w$; but not all of them are free. They are subject to two constraint relations $(2.1, 2.2)$ reducing the degrees of freedom down to two. Note in passing that by setting $y = \bar{x}$ and $w = \bar{z}$, the above relations reduce to

$$|x|^2 + |z|^2 = \text{Re} \mu,$$

$$(x, z) \rightarrow e^{i\theta} (x, z), \quad (2.3)$$

so they define a real two sphere $S^2$ embedded in complex space $\mathbb{C}^2$ parameterized by $(x, z)$. This is an interesting property valid not only for $T^*S^2$; but also for conifold $T^*S^3$. This crucial property will be used to recover the hermitian models on real three dimension space; it deals with the derivation of Lagrangian sub-manifold from mother manifold $T^*S^3$. As we will see it progressively, this feature is present everywhere along all of this paper. We will then keep it in mind and figure it out only when needed to make comments.
To implement string dynamics, we should add time variable $t$ and the string variable $\sigma$ parameterizing the one dimensional D string geometry. If we were dealing with a point like particle moving on this complex surface, the variables would be given by the 1d fields,

$$x = x(t), \quad y = y(t), \quad z = z(t), \quad w = w(t). \quad (2.4)$$

For the case of a D string with world sheet variable $\xi = t + i\sigma$ moving on $T^*P^1$, the D string variables are then given by the 2d fields,

$$x = x(\xi), \quad y = y(\xi), \quad z = z(\xi), \quad w = w(\xi). \quad (2.5)$$

with $|\sigma| \leq l$ and obviously the constraint eqs(2.1-2.2). In the limit $l \to 0$, the above 2d fields reduces to the previous one dimensional variables. Since K3 surface as considered here is a projective algebraic surface using complex holomorphic variables, it is natural to make the two following hypothesis:

(i) **Field Holomorphy:** We suppose that the above D-string field variables eqs(2.5) have no $\xi$ dependence; that is holomorphic functions in $\xi$,

$$\frac{\partial \phi}{\partial \xi} = 0, \quad \phi(\xi) = \sum_{n \in \mathbb{Z}} \alpha_n^\phi \xi^n, \quad \phi = x, y, z, w, \quad (2.6)$$

where $\alpha_n^\phi = \frac{1}{2\pi i} \oint \frac{d\xi}{\xi^{n+1}} \phi(\xi)$ are string modes. This hypothesis means that the D string we are dealing with is either a one handed mover closed D-string, say a left mover closed string, or an open D-string with free ends. To fix the ideas, we consider here below closed D-strings and think about $\xi = \exp(\tau + i\tilde{\sigma})$ with $0 \leq \sigma = l\tilde{\sigma} \leq 2\pi l$. Holomorphy hypothesis selects one sector; it requires that the variables parameterizing the D-strings are complex holomorphic and same for the field action $S_{DSF} = S_{DSF} [x, y, z, w]$ that describe their dynamics. Usual hermiticity is recovered by restricting conifold to its Lagrangian sub-manifold obtained by setting $\xi = \bar{\xi}, \ y = \bar{x}$ and $w = \bar{z}$.

(ii) **Induced gauge symmetry:** For later use it is interesting to treat on equal footing the string world sheet variable $\xi$ and those parameterizing K3. This may be done by thinking about the projective transformations (2.2) also as those one gets by performing the change,

$$\xi \to \lambda \xi, \quad (2.7)$$

with $\lambda$ a non zero complex parameter. In other words, the string variables obey the following,

$$x(\lambda \xi) = \lambda x(\xi), \quad z(\lambda \xi) = \lambda z(\xi), \quad (2.8)$$

$$y(\lambda \xi) = \frac{1}{\lambda} y(\xi), \quad w(\lambda \xi) = \frac{1}{\lambda} w(\xi), \quad (2.8)$$
together with the local constraint eqs,

\[ x(\xi)y(\xi) - z(\xi)w(\xi) = \mu. \]  \hspace{1cm} (2.9)

Note that eq(2.9) describes in fact an infinite set of constraint relations since for each value of \( \xi \in \mathbb{C}^* \), the D-string fields should obey (2.9). This feature has a nice geometric interpretation. The string dynamics involves five complex holomorphic variables namely \((\xi, x, y, z, w)\) and the two algebraic constraint equations (2.1-2.2). Therefore these variables parameterize a complex three dimension projective hypersurface embedded in \( \mathbb{C}^5 \) and which is nothing else that the deformed conifold geometry \( T^*S^3 \) with the realization,

\[ T^*S^3 \simeq T^*S^1 \times T^*S^2. \]  \hspace{1cm} (2.10)

In this fibration, \( T^*S^2 \) is the base sub-manifold and the fiber \( T^*S^1 \) describes the D-string world sheet.

To summarize, the variables describing the motion of a D-string in conifold are given by eqs(2.7-2.9). For a system of \( N \) D-strings moving in conifold, we have then,

\[ x_a(\xi)y_a(\xi) - z_a(\xi)w_a(\xi) = \mu, \quad a = 1, ..., N, \]  \hspace{1cm} (2.11)

where for each value of the index \( a \), we have also the eqs (2.7-2.8). Having fixed the variables, we turn now to describe the fluid approximation of D-strings and implement the constant and strong background RR B-field.

### 2.2 Fluid approximation

For later analysis, it is convenient to use the usual \( SL(2) \) isometry of the conifold to put the above relations into a condensed form. Setting

\[ X^i = (x(\xi), z(\xi)), \quad Y_i = (y(\xi), w(\xi)), \]  \hspace{1cm} (2.12)

transforming as isodoublets under \( SL(2) \) isometry, the coordinates of a given D string moving in conifold is given by the holomorphic field doublets,

\[ X^i = X^i(\xi), \quad Y_i = Y_i(\xi), \quad i = 1, 2, \]  \hspace{1cm} (2.13)

with the local constraint eqs,

\[ \epsilon_{ij}X^i(\xi)Y^j(\xi) = \mu, \]  \hspace{1cm} (2.14)

and the projective symmetry

\[ X^i(\lambda\xi) = \lambda X^i(\xi), \]
\[ Y_i(\lambda\xi) = \frac{1}{\lambda} Y_i(\xi). \]  \hspace{1cm} (2.15)
Using these notations, the system of D string reads then as follows

\[ \epsilon_{ij} X^i_a(\xi) Y^j_a(\xi) = \mu, \quad a = 1, \ldots, N, \quad (2.16) \]

where \( \epsilon_{ij} \) is the usual two dimensional antisymmetric tensor with \( \epsilon_{12} = 1 \). In the large \( N \) limit with density \( \rho(\xi, x, y) \); i.e \( N = \int_{T^*S^3} d^3\nu \rho(\xi, x, y) \), where \((x, y)\) sometimes denoted also as \((x^1, x^2, y_1, y_2)\) stand for the pairs of doublets \((X^i, Y^j)\), the D-string system may be thought of as a fluid of D1 branes running in conifold. Along with the previous relations, the fluid approximation allows the following substitutions,

\[ \{ X^i_a(\xi), 1 \leq a \leq N \} \rightarrow X^i(\xi, x, y), \]
\[ \{ Y^i_a(\xi), 1 \leq a \leq N \} \rightarrow Y^i(\xi, x, y), \quad (2.17) \]

together with eqs(2.16) replaced by

\[ \epsilon_{ij} X^i Y^j = \mu, \quad X^i = X^i(\xi, x, y), \quad Y^i = Y^i(\xi, x, y), \quad (2.18) \]

and projective symmetry promoted to,

\[ X^i \left( \lambda \xi, \lambda x, \frac{1}{\lambda} y \right) = \lambda X^i(\xi, x, y), \]
\[ Y^i \left( \lambda \xi, \lambda x, \frac{1}{\lambda} y \right) = \frac{1}{\lambda} Y^i(\xi, x, y). \quad (2.19) \]

For physical interpretation, we will also use the splitting

\[ X^i = x^i + \mu C^i_+, \quad Y_i = y_i - \mu C^-_i, \quad (2.20) \]

where \( C^i_+ \) and \( C^-_i \) are gauge fields constrained as

\[ x^i C^-_i - y_i C^i_+ + \mu C^-_i C^i_+ = 0, \quad (2.21) \]

scaling as the inverse of length and describing fluctuations around the static positions \( x^i \) and \( y_i \). From SL(2) representation theory, one may also split the fields \( X^i \) and \( Y^i \) using holomorphic vielbein gauge fields,

\[ X^i(\xi, x, y) = x^i E_{++} + \epsilon^{ij} y_j A_{++}, \]
\[ Y_i(\xi, x, y) = y_i E_{--} - \epsilon_{ij} x^j A_{--}, \quad (2.22) \]

where \( E_{\pm\mp} \) should be as \( E_{\pm\mp} = (1 + A_{\pm\mp}) \). Like for \( X^i \) and \( Y^i \), the gauge fields \( C^i_+ \) and \( C^-_i \) as well as \( E_{\pm\mp} \) and \( A_{\pm\mp} \) are homogeneous holomorphic functions subject to the
projective transformations $C_\pm (\lambda \xi, \lambda x, \frac{1}{\lambda} y) = \lambda^\pm C_\pm (\xi, x, y)$ and,

$$E_{+-} \left( \lambda \xi, \lambda x, \frac{1}{\lambda} y \right) = E_{+-} (\xi, x, y),$$
$$A_{++} \left( \lambda \xi, \lambda x, \frac{1}{\lambda} y \right) = \lambda^2 A_{++} (\xi, x, y),$$
$$E_{-+} \left( \lambda \xi, \lambda x, \frac{1}{\lambda} y \right) = E_{-+} (\xi, x, y),$$
$$A_{-+} \left( \lambda \xi, \lambda x, \frac{1}{\lambda} y \right) = \lambda^{-2} A_{-+} (\xi, x, y).$$

(2.23)

Using the conifold defining relation $\epsilon_{ij} \mathcal{X}^i \mathcal{Y}^j = \mu$, we see that, like for $C_\pm$ gauge fields, the above holomorphic vielbeins capture two complex degrees of freedom only since in addition to eqs (2.23), they satisfy moreover,

$$E_{+-} E_{-+} - A_{++} A_{-+} = 1. \quad (2.24)$$

An equivalent relation using $A_{\pm\mp}$ and $A_{\pm\pm}$ may be also written down. As far as the constraint eq (2.24) are concerned, there are more than one way to deal with. One way is to solve it perturbatively as $E_{+-} \approx (1 + A_{+-})$ with $A_{+-} = \pm A_0$ and then substitute $A_0 = i \sqrt{A_{++} A_{-+}}$. An other way is to solve eq (2.24) exactly as

$$E_{+-} = K \sqrt{1 + A_{++} A_{-+}},$$
$$E_{-+} = \frac{1}{K} \sqrt{1 + A_{++} A_{-+}},$$

(2.25)

where $K$ is an arbitrary non zero function. In both cases one looses field linearity which we would like to have it. We will then keep the gauge field constraint eqs as they are and give the results involving all these components using Lagrange method. Notice that, from physical view, the gauge fields $C_\pm$ or equivalently $A_{\pm\mp} = A_{\pm\mp} (\xi, x, y)$ and $A_{\pm\pm} = A_{\pm\pm} (\xi, x, y)$ describe gauge fluctuations around the static solution

$$\mathcal{X}^i = x^i, \quad \mathcal{Y}_i = y_i, \quad x^i y_i = \mu,$$

(2.26)

preserving conifold volume 3-form. Expressing the field $\mathcal{X}^i$ and $\mathcal{Y}_i$ as $\mathcal{X}^i = x^i + \mu C_+^i$ and $\mathcal{Y}_i = y_i - \mu C_-^i$, we have $\mu C_+^i = x^i A_{++} + y^i A_{++}$ and $\mu C_-^i = -y_i A_{++} + x_i A_{-+}$. Notice also that, as general coordinate transformations, the splitting (2.22) may be also defined as holomorphic diffeomorphisms $\mathcal{X}^i = \mathcal{L}_v x^i$ and $\mathcal{Y}_i = \mathcal{L}_v y_i$ where the vector field $\mathcal{L}_v$ is given by

$$\mathcal{L}_v = V_{+++} D_{++} + V_{++} D_{-+} + V_{+} D_{0} + V_{0}' \Delta_0,$$

(2.27)

with gauge component fields $V_{pq}$, $p, q = +, -, -$ and where the dimensionless derivatives generating the $GL(2)$ group are given by,

$$\Delta_0 = \frac{1}{2} \left( x^i \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial y^i} \right),$$

(2.28)
or naively as $\Delta_0 = \frac{\partial}{\partial (x^i y^i)}$, and

$$D_- = y^i \frac{\partial}{\partial x^i}, \quad D_+ = x^i \frac{\partial}{\partial y^i}, \quad D_0 = \left( x^i \frac{\partial}{\partial x^i} - y^i \frac{\partial}{\partial y^i} \right). \quad (2.29)$$

In these eqs, we have two charge operators; the operator $\Delta_0$ generates the abelian scaling factor with the property

$$[\Delta_0, D_{\pm\pm}] = 2D_{\pm\pm}, \quad [\Delta_0, D_0] = 0, \quad (2.30)$$

and $D_0 = [D_{++}, D_{--}]$ generates the abelian Cartan Weyl $GL(1)$ subgroup of $SL(2)$. Notice moreover that inverting the decomposition (2.22), we can write the vielbein fields as follows,

$$E_{+-} = \frac{1}{\mu} y_i x^i = 1 + y_i C^i_+, \quad E_{-+} = \frac{1}{\mu} y_i x^i = 1 - x^i C_{-i},$$

$$A_{++} = \frac{1}{\mu} \epsilon_{ij} x^i \mathcal{X}^j = -x^i C^i_+, \quad A_{--} = \frac{1}{\mu} \epsilon_{ij} y_j \mathcal{Y}^i = -y^i C_{-i}, \quad (2.31)$$

As one sees, these gauge fluctuations $E_{\pm\mp}$ and $A_{\pm\pm}$ are dimensionless; they let understand that they should appear as gauge fields covariantizing dimensionless linear differential operators. These are just the $D_{0,\pm\pm}$ operators given above. At the static point eq(2.26), we also see that $E_{+-} = E_{-+} = 1$ and $A_{\pm\pm} = 0, (C^i_\pm = 0)$. With these tools we are now in position to address the building of the effective field action of the D string fluid model in conifold.

3 Field action

To get the gauge invariant effective field action $S_{DSF} = S_{DSF} \left[ C^i_\pm, C_0 \right]$ describing the dynamics of the D string fluid in the conifold, we borrow ideas from Susskind method used for FQH liquid of point like particles. We first give the classical field action $S_{clas} [X, Y]$ describing the interaction between a given D string $\{X (\xi), Y (\xi)\}$ moving in the RR background field $B$. Then we consider the fluid approximation using the field variables $\{\mathcal{X} (\xi, x, y), \mathcal{Y} (\xi, x, y)\}$ instead of the coordinates $\{X_a (\xi), Y_a (\xi), \ 1 \leq a \leq N\}$. In this limit we suppose that density $\rho (\xi, x, y)$ is large and uniform; i.e $\rho (\xi, x, y) = \rho_0$. Finally, we derive the effective gauge field action once by using the D-string field variables $\mathcal{X}$ and $\mathcal{Y}$; i.e $S = S_{DSF} [\mathcal{X}, \mathcal{Y}, \ast]$ and an other time by using gauge fields $C^i_\pm$ and $C_0$ describing the fluctuations around the static positions.

3.1 Classical $B_{RR}$-D string coupling

To start recall that the field action $S_{clas} [z, \bar{z}] = \int dt L_{clas} (z, \bar{z})$ describing the classical dynamics of a charged particle with coordinate positions $z = z (t)$ and $\bar{z} = \bar{z} (t)$, in
a constant and strong background magnetic field $B$, is given by

$$L_{\text{clas}} = \frac{iB}{2} \left( \overline{\bar{z}(t)} \frac{d\bar{z}(t)}{dt} - z(t) \frac{d\bar{z}(t)}{dt} \right). \quad (3.1)$$

For a system of $N$ classical D-strings $\{X_a(\xi), Y_a(\xi), \quad 1 \leq a \leq N\}$ in the RR background magnetic field, one has a quite similar quantity. The above point like particle action extends as follows,

$$S_N [X, Y] = \frac{1}{2} \int_{T^*S^1} d\xi \sum_{a=1}^N B_{ij} \left( Y_a^j(\xi) \frac{\partial X_a^i(\xi)}{\partial \xi} - Y_a^i(\xi) \frac{\partial X_a^j(\xi)}{\partial \xi} \right), \quad (3.2)$$

with $B_{ij} = i\epsilon_{ij}$ and which, for convenience, we rewrite also as

$$S_N [X, Y] = \frac{iB}{2} \int_{T^*S^1} d\xi \sum_{a=1}^N \left( Y_a^i(\xi) \frac{\partial X_a^i}{\partial \xi} - X_a^i \frac{\partial Y_a^i}{\partial \xi} \right). \quad (3.3)$$

This field action $S_N [X, Y]$ exhibits three special and remarkable features; first it is holomorphic and the corresponding hermitian $S_N^{\text{real}} [X, \overline{X}]$ follows by setting,

$$Y_{ia} = \overline{(X^i_a)}, \quad \xi = \bar{\xi} = t, \quad B = \overline{B}. \quad (3.4)$$

As such we have

$$S_N^{\text{real}} [X, \overline{X}] = \frac{i \text{Re} B}{2} \int_{T^*S^1} dt \sum_{a=1}^N \left( \frac{\overline{(X^i_a)}}{\partial \xi} d\overline{(X^i_a)} - X_a^i \frac{\partial (\overline{X^i_a})}{\partial \xi} \right). \quad (3.5)$$

The second feature of $S_N [X, Y]$ deals with the hypersurface eq(2.16). Since $Y_{ia}X_a^i = \mu$ is a constraint eq on the dynamical field variables, it can be implemented in the action by using a Lagrange gauge field $\Lambda = \Lambda (\xi)$. So eq(3.2) should be read as,

$$S_N [X, Y, \Lambda] = \frac{iB}{2} \int_{T^*S^1} d\xi \sum_{a=1}^N \left( Y_{ia} \frac{\partial X_a^i}{\partial \xi} - X_a^i \frac{\partial Y_{ia}}{\partial \xi} \right)$$

$$+ \frac{B}{2} \int_{T^*S^1} d\xi \sum_{a=1}^N \Lambda_a(\xi) \left( Y_{ia}(\xi) X_a^i(\xi) - \mu \right). \quad (3.6)$$

The difference between $S_N [X, Y]$ of eq(3.3) and the above $S_N [X, Y, \Lambda]$ is that in the second description the field variables $X_a^i(\xi)$ and $Y_a^j(\xi)$ are unconstrained. Conifold target hypersurface is obtained by minimizing $S_N [X, Y, \Lambda]$ with respect to $\Lambda$,

$$\frac{\delta S_N [X, Y, \Lambda]}{\delta \Lambda_a} = Y_{ia}(\xi) X_a^i(\xi) - \mu = 0. \quad (3.7)$$

The third feature concerns the computation of the conjugate momentum $\Pi_i = \frac{\partial L}{\partial (\partial X^i/\partial \xi)}$ of the field variable $X^i$. One discovers that the coordinate variables $Y_i$ and $X^i$ are
conjugate fields. This property shows that the underlying conifold geometry with the background field behaves as a non commutative manifold.

Notice that, as required by the construction, \( \text{eq}(3.2) \) is invariant under the global symmetry

\[
\begin{align*}
\xi & \to \lambda \xi, \\
X^i_a & \to \lambda X^i_a, \\
Y^i_a & \to \frac{\Lambda}{\lambda} Y^i_a,
\end{align*}
\]

with \( \frac{d\lambda}{d\xi} = 0 \). This is a crucial point as far as we are thinking about conifold as given by the fibration \( T^* S^1 \times T^* S^2 \). Now, using the fluid approximation mapping the system \( \{ X^i_a(\xi), Y^j_a(\xi), \Lambda_a(\xi); 1 \leq a \leq N \} \) into the 3D holomorphic fields \( X^i = X^i(\xi, x, y) \), \( Y^j = Y^j(\xi, x, y) \) and \( * = * (\xi, x, y) \), we can put \( \text{eq}(3.6) \) as a complex 3D holomorphic field action

\[
S_2 [X, Y, *] = \int_{T^* S^3} d^3 v L_2 (X, Y, *),
\]

with

\[
L_2 (X, Y, *) = \frac{i B}{2 \mu} \left[ (Y_i \partial_0 X^i - X^i \partial_0 Y_i) - i \Lambda (X^i X^i - \mu) \right],
\]

and \( \partial_0 = \xi \frac{\partial}{\partial \xi} = \frac{\partial}{\partial \ln \xi} \) and where \( d^3 v \) is the conifold holomorphic volume measure given by,

\[
d^3 v = \frac{d\xi \wedge dx^i \wedge dy_i}{\xi}, \quad x^i y_i = \mu.
\]

For more details on the specific properties of this complex volume see [12]; for the moment let us push forward this description using the \( T^* S^1 \times T^* S^2 \) realization of conifold. In this view, notice that on \( T^* S^1 \), the global holomorphic operator \( \partial = d\xi \frac{\partial}{\partial \xi} \) may be also written as \( \partial = d\xi_0 \frac{\partial}{\partial \xi_0} \) with \( d\xi_0 = \frac{d\xi}{\ln \xi} \) and \( \partial_0 \) as before. Notice moreover that one can express the field action \( S_2 [X, Y, *] \) in term of the \( C_{\pm} \) gauge field fluctuations. Using the splitting \( X^i = x^i + \mu C^i_+ \) and \( Y^i = y_i - \mu C^-_i \), we obtain

\[
L_2 (C_{\pm}, \Lambda) = \frac{B \mu}{2i} \left[ (C_{-i} \partial_0 C^i_+ - C^i_+ \partial_0 C_{-i}) + i \Lambda \left( y_i C^i_+ - C^-_i x^i - C^-_i C^i_+ \right) \right],
\]

where we have dropped out the total derivatives \( \frac{d}{d\xi} \left( y_i C^i_+ + x^i C^-_i \right) \). Doing the same thing for the splitting \( X^i = x^i E_{+-} + y^i A_{++} \) and \( Y^i = y_i E_{--} - x_i A_{--} \) and substituting these relations back into \( \text{eq}(3.10) \), we get,

\[
L_2 \left[ E, A, \tilde{\Lambda} \right] = \frac{\mu^2 B}{2} (E_{+-} \partial_0 E_{+-} - E_{+-} \partial_0 E_{+-})
\]

\[
\quad + \frac{\mu^2 B}{2} (A_{--} \partial_0 A_{++} - A_{++} \partial_0 A_{--})
\]

\[
\quad + \frac{\mu B}{2} \tilde{\Lambda} (E_{+-} E_{--} - A_{++} A_{--} - 1),
\]

\( \text{eq}(3.13) \).
invariant under the projective symmetry with \( \tilde{\Lambda} \) a Lagrange gauge field parameter carrying the conifold constraint hypersurface. By using the \( D_0 \) charge operator, the transformations \( (2.23) \) can be also stated as \( D_0 E_{\pm} = 0, D_0 A_{\pm} = \pm 2 A_{\pm} \); they follow as well from the identities \( D_0 \mathcal{X}^i = \mathcal{X}^i \) and \( D_0 \mathcal{Y}^i = -\mathcal{Y}^i \). Note that by substituting \( E_{+} = 1 + A_{+} \) and \( E_{-} = 1 - A_{-} \), one sees that the term \( (E_{-} \partial_0 E_{+} - E_{+} \partial_0 E_{-}) \) reduces to a total derivative \( \partial_0 (2A_{+-}) \) and so can be ignored in such a realization.

3.2 Implementing density constraint equation

First note that to get the density constraint eq in the fluid approximation, one computes the total number \( N \) of D strings by using two paths; once by the coordinate frame \( \{ x, y \} \) and second by using the frame \( \{ \mathcal{X}, \mathcal{Y} \} \). Then equating the two expressions since this number is invariant under coordinate transformation. Supposing that fluid density is uniform \( \rho (\xi, x, y) = \rho_0 \), a property implying,

\[
N = \int_{T^* S^3} \rho d^3 v = \rho_0 \int_{T^* S^3} d^3 v, \quad (3.14)
\]

and using the fact that this number is a constant, one gets a constraint eq on the Jacobian \( J (x, y) = \left| \frac{\partial (\mathcal{X}, \mathcal{Y})}{\partial (x, y)} \right| \) of the general transformation,

\[
x \rightarrow \mathcal{X} = \mathcal{X} (x, y), \quad y \rightarrow \mathcal{Y} = \mathcal{Y} (x, y). \quad (3.15)
\]

Eq \( (3.14) \) requires that \( J (x, y) = 1 \). Let us give some details on this calculation. Since the density is uniform, we should have

\[
\rho_0 \int_{T^* S^3} d^3 \mathcal{V} = \rho_0 \int_{T^* S^3} d^3 v, \quad (3.16)
\]

Using the explicit expressions of the conifold holomorphic volume 3-form which we write first as \( d^3 \mathcal{V} = \frac{d \xi}{\xi} \wedge d^2 S \) and second \( d^3 v = \frac{d \xi}{\xi} \wedge d^2 s \). Then expanding the K3 holomorphic 2-form \( d^2 S = (d \mathcal{X}^i \wedge d \mathcal{Y}_i) \), we get after some straightforward algebra,

\[
\mu d^2 S = \left\{ \mathcal{X}^i, \mathcal{Y}_i \right\}_{++} d^2 s \quad (3.17)
\]

\[
\quad + \left\{ \mathcal{X}^i, \mathcal{Y}_i \right\}_{0-} dx^i \wedge dx_i + \left\{ \mathcal{X}^i, \mathcal{Y}_i \right\}_{0+} dy^i \wedge dy_i.
\]

In this relation \( d^2 s = (dx^i \wedge dy_i) \) and \( \{ f, g \}_{p,q} \) stand for the Poisson brackets defined as,

\[
\{ f, g \}_{++} = (D_{++} f) (D_{--} g) - (D_{--} f) (D_{++} g),
\]

\[
\{ f, g \}_{0-} = (D_0 f) (D_{--} g) - (D_{--} f) (D_0 g),
\]

\[
\{ f, g \}_{0+} = (D_0 f) (D_{++} g) - (D_{++} f) (D_0 g),
\]

with \( D_{\pm \pm, 0} \) generating the \( SL (2, \mathbb{C}) \) isometry eqs \( (2.29) \). Volume preserving diffeomorphisms require then the following constraint eqs to be hold,

\[
\left\{ \mathcal{X}^i, \mathcal{Y}_i \right\}_{++} = (D_{++} \mathcal{X}^i) (D_{--} \mathcal{Y}_i) - (D_{--} \mathcal{X}^i) (D_{++} \mathcal{Y}_i) = \mu, \quad (3.19)
\]
and

\[ \{ X^i, Y_i \}_{0-} = (D_0 X^i) (D_- Y_i) - (D_- X^i) (D_0 Y_i) = 0, \quad (3.20) \]
\[ \{ X^i, Y_i \}_{0+} = (D_0 X^i) (D_{++} Y_i) - (D_{++} X^i) (D_0 Y_i) = 0. \]

A careful inspection shows that the last two conditions are not really constraint eqs. The point is that because of the identities,

\[ D_0 X^i = X^i, \quad D_0 Y_i = -Y_i, \quad (3.21) \]

required by K3 geometry, the two last constraint eqs can be brought to,

\[ \{ X^i, Y_i \}_{0-} = D_- (X^i Y_i), \quad \{ X^i, Y_i \}_{0+} = D_{++} (X^i Y_i). \quad (3.22) \]

But these relations vanishes identically because of the identity \( X^i Y_i = \mu = \text{constant} \). Therefore the volume transformation (3.17) becomes \( \mu d^2 S = \{ X^i, Y_i \}_{++} d^2 s \) and so we are left with one constraint relation; namely \( \{ X^i, Y_i \}_{++} = \mu \) which can be implemented in the field action (3.10) by help of a Lagrange gauge field \( C_0 \). To that purpose note that by setting \( J_{\pm \pm} = \pm (C_0 Y_i D_{\pm \pm} X^i) \), one can check that we have,

\[ \int d^3 v C_0 \left[ \{ X^i, Y_i \}_{++} - \mu \right] = \int d^3 v \left( Y_i \{ C_0, X^i \}_{++} \right) \quad (3.23) \]

where we have dropped out the boundary term \( \int d^3 v \left[ D_- J_{++} + D_{++} J_{--} \right] \). Implementing this identity in the field action as usual, we get the following holomorphic functional

\[ S_{DSF} [X, Y, C_0] = \int_{T^* S^3} d^3 v \quad L_{DSF} (X, Y, C_0), \quad (3.24) \]

with,

\[ L_{DSF} [X, Y, C_0] = \frac{iB}{2\mu} \left( Y_i \partial_0 X^i - X^i \partial_0 Y_i \right) + \frac{B}{2\mu} \Lambda \left( Y_i X^i - \mu \right) \]
\[ -\frac{B}{\mu} \left( Y_i \{ C_0, X^i \}_{++} - X^i \{ C_0, Y_i \}_{++} \right) \quad (3.25) \]

Using the previous splitting of the D string fields \( X^i \) and \( Y_i \), we can express this field action in terms of the gauge fields either as \( S_{DSF} = S_{DSF} [C_{\pm \pm}, C_0, \Lambda] \) or equivalently as \( S_{DSF} = S_{DSF} [E, A, C_0, \Lambda] \). Let us do this calculation for the splitting \( X^i = x^i + \mu C^i_+ \) and \( Y_i = y_i - \mu C_{-i} \). In this case the density constraint eq \( \{ X^i, Y_i \}_{++} = \mu \) reads in terms of the \( C_{\pm i} \) gauge fields as follows,

\[ \{ x^i, C_{-i} \}_{--} - \{ C^i_+, y_i \}_{--} + i\mu \{ C^i_+, C_{-i} \}_{--} = 0. \quad (3.26) \]

This relation can be put into a more interesting way by setting \( \{ x^i, F \}_{--} = \partial_+ F, \{ F, y_i \}_{--} = \partial_- F \) with the remarkable properties \( \partial_+ \partial_- = -y_i D_{++} (x^i D_{--}) = -\mu D_{++} D_{--} \)
and \( \partial_{-i} \partial^i_+ = -x^i D_{-} (y_i D_{++}) = -\mu D_{-} D_{++} \). Putting these relations back into (3.26), we obtain

\[
\partial^i_+ C_{-i} - \partial_{-i} C^i_+ - i \left( \partial^k_+ C^i_+ \partial_{-k} C_{-i} - \partial_{-k} C^i_+ \partial^k_+ C_{-i} \right) = 0, \quad (3.27)
\]

or equivalently by introducing Poisson bracket \( \{ F, G \}_PB \equiv (\partial_{+k} F) \left( \partial^k_- G \right) - (\partial^k_- F) \left( \partial_{+k} G \right) \),

\[
\partial^i_+ C_{-i} - \partial_{-i} C^i_+ - i \left\{ C^i_+, C_{-i} \right\}_PB = 0. \quad (3.28)
\]

Note also that \( \{ F, G \}_PB \) is just \( \mu \{ F, G \}_{-+} \). As we see, this is a typical equation of motion of non-commutative gauge theory; it can be then thought of as the minimization of an invariant gauge field \( S_{DSF} [C_{\pm}, C_0] \) with gauge fields \( C^i_{\pm} \) and \( C_0 \). In this view, we have,

\[
\frac{\delta S_{DSF} [C_{\pm}, C_0]}{\delta C_0} = \partial^i_+ C_{-i} - \partial_{-i} C^i_+ - i \left\{ C^i_+, C_{-i} \right\}_PB = 0, \quad (3.29)
\]

from which we can determine \( S_{DSF} [C_{\pm}, C_0] \) taking into account eq(3.12). Setting

\[
S_{DSF} [C^i_{\pm}, C_0, \Lambda] = \int B \int_{T^* S^3} d^3v L_{DSF} [C_{\pm}, C_0, \Lambda],
\]

we have

\[
L_{DFS} [C_{\pm}, C_0, \Lambda] = \frac{i B \mu}{2} \left( C^i_+ \partial_0 C_{-i} - C_{-i} \partial_0 C^i_+ \right) + \\
- \frac{B \mu}{2} \left[ 2 \left( C_0 \partial^i_+ C_{-i} - C_{-i} \partial_0 C^i_+ \right) - 2 C_0 \left\{ C^i_+, C_{-i} \right\}_PB \right] + \frac{B \mu}{2} \Lambda \left( y_i C^i_+ - C_{-i} x^i - C_{-i} C^i_+ \right). \quad (3.30)
\]

This holomorphic lagrangian density may be put into a more convenient way by performing an integration by part and dropping out the total derivatives. Replacing

\[
C_0 \{ F, G \}_{-+} = -F \{ C_0, G \}_{-+} + FC_0 D_0 G + \text{total derivative} \quad (3.31)
\]

for holomorphic functions \( F \) and \( G \) on conifold we have,

\[
L_{DSF} [C_{\pm}, C_0] = \frac{i B \mu}{2} \left[ C^i_+ \partial_0 C_{-i} - C^i_{-i} \partial_{-i} C_0 - \frac{2i}{3} C^i_+ \left\{ C_0, C_{-i} \right\}_PB \right] + \\
+ \frac{i B \mu}{2} \left[ -C_0 \partial^i_+ C_{-i} + C_0 \partial_{-i} C^i_+ + \frac{2i}{3} C_0 \left\{ C^i_+, C_{-i} \right\}_PB \right] + \frac{i B \mu}{2} \left[ C_{-i} \partial^i_+ C_0 - C_{-i} \partial_0 C^i_+ - \frac{2i}{3} C_{-i} \left\{ C^i_+, C_0 \right\}_PB \right]. \quad (3.32)
\]

where we have set \( (y_i C^i_+ - x^i C_{-i} - C^i_{-i} C_{-i}) = 0 \) describing gauge fluctuations restricted to conifold. By substituting \( \mu C^i_+ = x^i A_{++} + y^i A_{++} \) and \( \mu C_{-i} = -y_i A_{-+} + x_i A_{-+} \) in the above gauge field action, one gets the expression of \( L_{DSF} [A_{++}, A_{++}, A_{-+}, A_{-+}] \) in terms of the gauge fields \( A_{++}, A_{++}, A_{-+} \) and \( A_{-+} \).
In the end notice that on the real slice of conifold with parameter Re $\mu$, background field Re $B$ and field variables as,

$$\mathcal{Y}_i = \overline{(X^i)}, \quad \Leftrightarrow \quad C_{-i} = \overline{(C^i)},$$

the previous field action reduces to non commutative Chern Simons gauge theory in real three dimensions. In this case $(\text{Re} \, B) \times (\text{Re} \, \mu)$ should be equal to Kac-Moody level $k$.

## 4 Holomorphy and quantum corrections

Though natural from classical view, the correspondence between FQH systems and fluids of D-strings in conifold described above is however no longer obvious at quantum level. In the D-string fluid proposal, the classical free degrees of freedom of the holomorphic sector,

$$S_N [X, Y] = \frac{iB}{2} \int_{T^* S^3} d\zeta \sum_{a=1}^{N} \left( Y_{ia} \frac{\partial X^i_a}{\partial \zeta} - X^i_a \frac{\partial Y_{ia}}{\partial \zeta} \right),$$

and the corresponding antiholomorphic one,

$$S_N^* [X^*, Y^*] = -\frac{iB}{2} \int_{T^* S^3} d\zeta \sum_{a=1}^{N} \left( Y^*_{ia} \frac{\partial X^*_a}{\partial \zeta} - X^*_a \frac{\partial Y^*_{ia}}{\partial \zeta} \right),$$

may couple quantum mechanically unless this is forbidden by underlying symmetries. Typical examples of these powerful symmetries, one encounters in such kind of situations, are generally given by conformal invariance, supersymmetry and their extensions. In this section, we make general comments on quantum effects in the D string system and give a discussion on how supersymmetry can help to overcome difficulties. Implication of supersymmetry in the game can be motivated from several views starting from complex Kahler geometry of $T^* S^3$ and ending with topological aspects of $2d$ fields on conifold. To fix the ideas on the way we will do things, we recall the standard parallel between field holomorphy in conifold geometry and chirality in $2d \, \mathcal{N} = 2$ supersymmetric non linear sigma model captured by the usual supersymmetric derivatives $\mathcal{D}_{\pm 1/2}$. Using this parallel, we shall show that the holomorphic lagrangian density $L (X, Y) = B \sum_{a=1}^{N} Y_{ia} (\partial X^i_a / \partial \zeta)$ of the D-string fluid can be thought of as following from the chiral superspace lagrangian of the $\mathcal{N} = 2$ supersymmetric sigma model in large B field,

$$L_{\text{chiral}} [\Phi] = \int_{SM_{-}} d^{2} \theta \, \mathcal{W} (\Phi),$$

where $\Phi$ refers to generic chiral superfields and $SM_{-}$ to chiral superspace. In this relation, $\mathcal{W} (\Phi) \sim \left( B \sum_{a=1}^{N} \Phi_{a1} \Phi_{a2} \right)$ is chiral the superpotential. Substituting the chiral
superfields $\Phi_{ia}$ by their $\theta$-expansions; i.e

$$\Phi_{ia} \sim Y_{ia} + \ldots + \theta_{+1/2} \theta_{-1/2} F_{ia}, \quad i = 1, 2,$$

(4.4)

where we have dropped out fermions and where $F_{ia}$ are auxiliary fields to be specified in a moment; then integrating with respect to the Grassman variables $\theta_{\pm 1/2}$, gives the following field component product $B \left( \sum_{a=1}^{N} Y_{ia} F_{ia} \right)$. By taking the auxiliary fields $F_{ia}$ as,

$$F_{ia} = \left( \sqrt{\mu} \epsilon_{ij} X^j_a + \epsilon_{ij} \frac{\partial X^j_a}{\partial \zeta} \right),$$

(4.5)

where $\epsilon_{ij}$ is the usual spinor metric and $\mu$ the conifold complex parameters, one discovers, up to a constant, the above holomorphic lagrangian density.

Before going ahead, it should be also noted that the comments we shall give below are certainly not final answers; but just a tentative to approach aspects of quantum behaviour of D string fluid in conifold. The discussion presented below relies on path integral method for quantization. But may be the more natural way to do would be extending matrix model approach of Susskind-Polychronakos (SP) for FQH droplets. Recall that SP method uses canonical quantization. We will give a brief comment on this method in the end of this section. More involved details may be found in [15].

This discussion is organized as follows: In the first subsection, we explore the consequences of quantum effects on conifold geometry and derive the constraint eq on quantum consistency of holomorphy property. Using path integrals quantization method, we show that holomorphy persists as far quantum fluctuations are restricted to complex deformations of conifold. Implementation of Kahler deformations destroys this behaviour since holomorphic and antiholomorphic modes get coupled. In sub-section 2, we study the embedding the D string model in a supersymmetric theory and too particularly in its chiral sectors. The latter seems to be the appropriate theory that governs the quantum fluctuations of the D-string fluid in conifold. As a first step in checking this statement, we start by describing the field theoretic derivation of holomorphy hypothesis considered in section 2. Then we give a correspondence with $2d \mathcal{N} = 2$ supersymmetric non linear sigma model with conifold as a target space; in presence of a background magnetic field $B$. We end this section by discussing the statistics of the D-string system which requires a filling fraction $\nu = \frac{1}{k}$ with even integer Kac-Moody levels $k$.

4.1 Quantum effects and conifold deformations

A way\footnote{An other tentative to approach the fluid of D-strings in conifold, by using a generalization of matrix model method based on canonical quantization, has been developed in [15]. There and as a first step in} to study the quantum effects on the holomorphy feature of the D-string fluid model is to proceed as follows. First think about the D string fluid model as a...
classical field theory based on the conifold geometry $xy - zw = \mu$. This means that
the complex threefold, with its complex modulus $\mu$, can be thought of as a classical
geometry. Quantum mechanically, the above fields are subject to fluctuations and so
the complex parameter $\mu$ gets corrections induced by quantum effects. To have an idea
on the nature of these quantum corrections, we consider fluctuations of the D-strings
around the classical field configurations $x, y, z, w$. These field fluctuations can be written
as
\[ \phi \rightarrow \phi + \delta F_\phi, \quad \phi = x, y, z, w, \]  
(4.6)
with the generic fields $\phi = \phi(\xi)$ as in eq(2.6) and $\delta F_\phi$ describing the perturbations
around the classical field $\phi$. Notice that these fluctuations are involved in the computa-
tion of the partition function $Z[j]$ of the model,
\[ Z[j] = \int \left( \prod D\phi \right) \exp \frac{i}{\hbar} \left( S[\phi] + \int j \phi \right), \]  
(4.7)
where $\prod D\phi$ stays for the usual field path integral measure. As it is known, this quantity
generates the Green functions of the quantum model with $j$ being the usual external
source. Notice also that the $\delta F_\phi$ deformations should a priori depend on both the string
fields $\phi$ and their complex conjugates $\overline{\phi}$ as shown below,
\[ \delta F_\phi = F(\phi, \overline{\phi}). \]  
(4.8)
By implementing the fluctuations (4.6) into the D-string fluid model, one discovers that
the classical geometry $xy - zw = \mu$ we started with gets now deformed as follows,
\[ xy - zw = \mu + \mathcal{F}, \]  
(4.9)
where the functional $\mathcal{F}$ capturing the field fluctuations is given by,
\[ \mathcal{F} = x\delta F_y + y\delta F_x - z\delta F_w - w\delta F_z. \]  
(4.10)
Like for eq(4.8), one sees that $\mathcal{F}$ depends in general on both the fields $x, y, z, w$ and
their complex conjugate $\overline{x}, \overline{y}, \overline{z}, \overline{w}$,
\[ \mathcal{F} = \mathcal{F}(\phi, \overline{\phi}), \quad \phi = x, y, z, w. \]  
(4.11)
Thus, quantum mechanical effects encoded in the functional $\mathcal{F}(\phi, \overline{\phi})$ break holomorphy
of the classical conifold geometry unless field deformations $\delta F_\phi$ are restricted to holo-
morphic perturbations around the classical field configuration. In this special case, we have,
\[ \frac{\partial \mathcal{F}}{\partial \overline{\phi}} = 0; \]  
(4.12)
dealing with the problem, one focuses on the study of quantum droplets for the conifold sub-varieties
$S^3$ and $S^2$. 
and so classical holomorphy is preserved quantum mechanically. This is the condition for quantum decoupling of holomorphic and antiholomorphic degrees of freedom. This property has a geometric interpretation in term of conifold structure deformations; it means that only complex deformations of holomorphic volume that are allowed for having a consistent quantum mechanics. It is also interesting to note that eq (4.12) is a strong condition; its solution requires however a strong symmetry which apparently D-string fluid model does not exhibit manifestly; at least not as things have been formulated so far. Note moreover that as far as quantum holomorphy is concerned, to our knowledge only supersymmetry that has the magic power to deal with target space holomorphy. There, quantum corrections are controlled by the so called non renormalization theorem.

The next question is how the string fluid model could be related to 2\(d\) \(\mathcal{N} = 2\) supersymmetric non linear sigma model with conifold as target space. Thinking about the D-string model as the bosonic part of a supersymmetric theory does not answer exactly the question since there are Kahler deformations induced by quantum effects that destroy the classical holomorphy property. To overcome such difficulty one should then associate the action of the D-string model with chiral superpotentials,

\[
\mathcal{W} = \mathcal{W}(\Phi), \tag{4.13}
\]

of \(\mathcal{N} = 2\) supersymmetric non linear sigma model. In what follows, we develop a way to do it. Though not exact and needs more investigations, this approach offers however an important step towards the goal.

### 4.2 Supersymmetric embedding

To begin recall that there is a closed connection between Kahler geometry and \(\mathcal{N} = 2\) supersymmetry in two dimensions. The fact that the fluid of D-strings is described by a topological holomorphic gauge theory, let understand that this model can be embedded in a \(\mathcal{N} = 2\) supersymmetric theory; from which one can get informations about quantum corrections. In this view holomorphy property is interpreted as the target space manifestation of chirality feature of 2\(d\) \(\mathcal{N} = 2\) supersymmetric sigma models with conifold as target space. A close idea is used in building topological string theory by using twist of 2\(d\) \(\mathcal{N} = 2\) superconformal algebra \[17\] and a correspondence with type II superstrings on Calabi-Yau threefolds \[18\]. In our concern, we have the following correspondence,

\[
\begin{align*}
\int d\zeta \ldots & \rightarrow \int d^2 \theta \ldots, \\
\int d\bar{\zeta} \ldots & \rightarrow \int d^2 \bar{\theta} \ldots, \\
\int d^2 \zeta \ldots & \rightarrow \int d^2 \theta d^2 \bar{\theta} \ldots, \tag{4.14}
\end{align*}
\]
with the $\theta_{\pm 1/2}$'s and $\overline{\theta}_{\pm 1/2}$'s the usual Grassman variables. Similar things may be also written down for $\partial / \partial \zeta$ and supersymmetric derivatives. Before that, let us start by deriving rigorously the holomorphy hypothesis of section 2 by using a field theoretical method; then come back to the correspondence between target space holomorphy and 2d $\mathcal{N} = 2$ supersymmetric chirality.

### 4.2.1 Holomorphy property and boundary QFT$_2$

Holomorphy is one of the basic ingredients we have used in deriving the D-string model developed in this paper. It has been imposed in order to complete the conifold realization $T^*S^3$ as a fibration of $T^*S^1$ over the base $T^*S^2$. In this study, we first give the field theoretic derivation of this holomorphy hypothesis; it appears as the solution of a constraint eq required by boundary field theory in two dimensions. Then we derive the field action (4.15); its connection with supersymmetric models is considered in the next sub-subsection.

To proceed and seen that the model we are studying involves complex fields, it is then natural to start from the following bosonic QFT$_2$ field action,

$$S [\phi, \overline{\phi}] = \int_M d^2 \zeta \left( G_{\alpha \overline{\beta}} \partial_+ \phi^\alpha \partial_- \overline{\phi}^{\overline{\beta}} \right),$$

(4.15)

where $M$ is a real surface parameterized by the local complex coordinates $(\zeta, \overline{\zeta})$. The fields $\phi^\alpha = \phi^\alpha (\zeta, \overline{\zeta})$ form a set of complex 2d scalar fields parameterizing some target Kahler manifold with metric $G_{\alpha \overline{\beta}} = G_{\alpha \overline{\beta}} (\phi, \overline{\phi})$. To make contact with conifold geometry and the fluid of $N$ strings, we think about these field variables as,

$$\phi^\alpha (\zeta, \overline{\zeta}) = X^i_a (\zeta, \overline{\zeta}), \quad i = 1, 2, \quad a = 1, ..., N,$$

(4.16)

with $X^i_a$ an SU(2) doublet like in eq(2.12) and to fix the ideas the field doublet $Y_{ia}$ are set to $\overline{X}^i_a$. Once the idea is exhibited, the field $\overline{X}^i_a$ will be promoted to $Y_{ia}$. In this case, the Kahler metric $G_{\alpha \overline{\beta}}$ may be split as

$$G_{\alpha \overline{\beta}} (\phi, \overline{\phi}) = \delta_{ab} \left[ g^{ij} + B \epsilon^{ij} \right],$$

(4.17)

where the $SU(2)$ triplet $g_{ij}$ is a function on the target space field coordinates; i.e $g_{ij} = g_{ij} (\phi, \overline{\phi})$, and where $\epsilon^{ij}$ is the usual antisymmetric $SU(2)$ invariant tensor. In the special case where $B$ is field independent and strong enough so that we can neglect the term $g_{ij}$, the metric $G_{\alpha \overline{\beta}}$ reduces essentially to $B \delta_{ab} \epsilon^{ij}$; and so one is left with the following approximated field action,

$$S [X, \overline{X}] \simeq \int_M d^2 \zeta \left( B \epsilon_{ij} \sum_{a=1}^N \partial_+ X^i_a \partial_- X^i_a \right),$$

(4.18)
where we have set $\zeta = \zeta_-, \bar{\zeta} = \zeta_+$ and $\partial_+ = \partial_\zeta, \partial_- = \partial_{\bar{\zeta}}$. Moreover since $B$ is a constant, one can split this action as follows,

$$S \left[ X, \bar{X} \right] \approx \frac{B}{2} \int_M d\zeta_- \left[ d\zeta_+ \partial_- \left( \sum_{a=1}^N (\partial_+ \bar{X}_{ia}) X_a^i \right) \right] + \frac{B}{2} \int_M d\zeta_+ \left[ d\zeta_- \partial_+ \left( \sum_{a=1}^N \bar{X}_{ia} (\partial_- X_a^i) \right) \right]$$

(4.19)

$$- \frac{B}{2} \int_M d^2 \zeta \sum_{a=1}^N \left[ (\partial_- \partial_+ \bar{X}_{ia}) X_a^i + \bar{X}_{ia} (\partial_+ \partial_- X_a^i) \right],$$

where the summation over $SU(2)$ indices is understood. By integrating the two first terms of above relation, one sees that the field action $S \left[ X, \bar{X} \right]$ decomposes as,

$$S \approx S^{\text{boundary}} + S^{\text{bulk}}$$

(4.20)

with two factors for $S^{\text{boundary}} = S^{\text{bound}}$ as given below,

$$S^{\text{bound}} = \frac{B}{2} \int_{\partial M_-} d\zeta \left( \sum_{a=1}^N (\partial_+ \bar{X}_{ia}) X_a^i \right) + \frac{B}{2} \int_{\partial M_+} \bar{d}\zeta \left( \sum_{a=1}^N \bar{X}_{ia} (\partial_- X_a^i) \right),$$

(4.21)

where $\partial M_\pm$ stand for the oriented boundaries of the Riemann surface $M$ and

$$S^{\text{bulk}} = - \int_M d^2 \zeta \sum_{a=1}^N \left[ \frac{B}{2} (\partial_- \partial_+ \bar{X}_{ia}) X_a^i + \frac{B}{2} \bar{X}_{ia} (\partial_+ \partial_- X_a^i) \right].$$

(4.22)

Equating eq(4.18) and eq(4.21), one gets the holomorphy condition of the field variables,

$$\left[ \frac{\partial}{\partial \zeta} \frac{\partial}{\partial \bar{\zeta}} X_a^i (\zeta, \bar{\zeta}) \right]_{\partial M} = 0, \left[ \frac{\partial}{\partial \zeta} \frac{\partial}{\partial \bar{\zeta}} \bar{X}_{ia} (\zeta, \bar{\zeta}) \right]_{\partial M} = 0.$$  

(4.23)

These constraint relations are solved by field holomorphy as shown below;

$$X_a^i (\zeta, \bar{\zeta}) = X_a^i (\zeta) + X_a^i (\bar{\zeta}),$$

$$\bar{X}_{ia} (\zeta, \bar{\zeta}) = \bar{X}_{ia}^* (\zeta) + \bar{X}_{ia}^* (\bar{\zeta}).$$  

(4.24)

They tell us that on the boundary $\partial M$ of the Riemann surface, we have two heterotic free field theories; a holomorphic sector with field variables

$$X_a^i (\zeta), \quad X_{ia}^* (\zeta),$$

(4.25)

which, for convenience and avoiding confusion we set $X_{ia}^* (\zeta) = Y_{ia} (\zeta)$, and an antiholomorphic one with,

$$X_{ia}^* (\zeta) = \bar{X}_{ia}^* (\bar{\zeta}), \quad X_a^i (\bar{\zeta}) = \bar{Y}_{ia} (\bar{\zeta})$$  

(4.26)

in agreement with the hypothesis on holomorphicity of the string coordinates.
4.2.2 Supersymmetric interpretation

The decomposition of the field action \( S[QFT_2] \) eqs(4.15-4.18) taken in the limit large \( B \) field is very suggestive. First, because it explains the origin of holomorphy hypothesis we have used to build the model of the fluid of D strings. As such, one should distinguish between fields in bulk and fields in boundary given by eqs(4.25-4.26). Second it permits a one to one correspondence with \( 2d \mathcal{N} = 2 \) supersymmetric non linear sigma models. More precisely the three terms of the field action of the bosonic QFT\(_2\) in large B limit,

\[
S[QFT_2] = - \int_M d^2\zeta \sum_{a=1}^N \left[ \frac{B}{2} X^i_a (\partial_- \partial_+ X_{ia}) + \frac{B}{2} \bar{X}_{ia} (\partial_+ \partial_- X^i_a) \right] - \int_{\partial M_-} d\zeta \left( \frac{B}{2} \sum_{a=1}^N Y_{ia} (\partial_+ X^i_a) \right) - \int_{\partial M_+} d\bar{\zeta} \left( \frac{B}{2} \sum_{a=1}^N \bar{Y}_{ia} (\partial_- X^i_a) \right),
\]

are in one to one with the usual three blocks of \( 2d \mathcal{N} = 2 \) supersymmetric non linear sigma models,

\[
S_{\mathcal{N}=2} [\Phi, \Phi^+] = \int_{SM} d^2\nu d^2\theta d^2\bar{\theta} \mathcal{K} (\Phi_i, \Phi_i^+) + \int_{SM_-} d^2\nu d^2\theta \mathcal{W} (\Phi_i) + \int_{SM_+} d^2\nu d^2\bar{\theta} \bar{\mathcal{W}} (\Phi_i^+).
\]

In this relation, the symbol \( SM \) stands for the usual two dimensional superspace with super-coordinates \( (\nu, \theta_{\pm 1/2}, \bar{\theta}_{\pm 1/2}) \) and \( SM_{\pm} \) stand for the two associated chiral superspaces. The \( \Phi_i \)'s (resp. \( \Phi_i^+ \)) are chiral (resp. antichiral) superfields living on \( SM_- \) (resp. \( SM_+ \)), \( \mathcal{K} (\Phi, \Phi^+) \) is the Kahler superpotential and \( \mathcal{W} (\Phi) \) the usual complex chiral superpotential. Like for the holomorphic functions \( f = f(\zeta) \) living on \( \partial M_- \) and satisfying the holomorphy property,

\[
\frac{\partial f}{\partial \zeta} = 0,
\]

we have for chiral superfields \( \Phi (\bar{\nu}_{\pm}, \theta_{\pm 1/2}) \) living on \( SM_- \), the following chirality property,

\[
\bar{D}_{\pm 1/2} \Phi = 0.
\]

By comparison of the two actions, one sees that the bulk term \( S^{bulk} \) of the QFT\(_2\) eq(4.27) is associated with Kahler term of the supersymmetric sigma model,

\[
S^{bulk}[QFT_2] \longleftrightarrow \int_{SM} d^2\nu d^2\theta d^2\bar{\theta} \mathcal{K} (\Phi_i, \Phi_i^+),
\]
while the two boundary terms $S^\text{bound}_\pm$ are associated with the chiral superfield actions. More precisely, we have

$$\int_{\partial M_-} d\zeta \left( \sum_{a=1}^{N} \frac{B}{2} Y_i F_a^i \right) \longleftrightarrow \int_{S M_-} d^2 \nu d^2 \theta \mathcal{W}(\Phi_i), \quad (4.32)$$

where we have set $F_a^i = (\partial_{\zeta} X_a^i)$ and by putting after setting $F_a^i = (\partial_{\zeta} (X_a^i))$, we also have

$$\int_{\partial M_+} d\zeta \left( \sum_{a=1}^{N} \frac{B}{2} (Y_a (F_a^i)) \right) \longleftrightarrow \int_{S M_+} d^2 \nu d^2 \theta \mathcal{W}(\Phi_i^+). \quad (4.33)$$

Now, considering two chiral superfields $\Phi_1 = \Phi_1 (v_{\pm}, \theta_{\pm1/2})$ and $\Phi_2 = \Phi_2 (v_{\pm}, \theta_{\pm1/2})$ with $\theta$- expansions,

$$\Phi_1 = Y_1 + \theta_{+1/2} \psi_{+1/2} + \theta_{-1/2} \psi_{+1/2} - \theta_{+1/2} \psi_{-1/2} + \theta_{+1/2} \theta_{-1/2} F_1,$$

$$\Phi_2 = Y_2 + \theta_{+1/2} \varphi_{+1/2} + \theta_{-1/2} \varphi_{+1/2} - \theta_{+1/2} \varphi_{-1/2} - \theta_{+1/2} \theta_{-1/2} F_2,$$ \quad (4.34)

with $Y_i$ and $F_i$ being the bosonic complex fields, we can build the superpotential associated with the boundary QFT$_2$. We have,

$$\int_{S M_-} d^2 \theta \left( \sum_{a=1}^{N} \frac{B}{2} \Phi_{a1} \Phi_{a2} \right) = - \sum_{a=1}^{N} \left( \frac{B}{2} Y_{a1} F_{a2} - Y_{a2} F_{a1} \right), \quad (4.35)$$

which can be also written a covariant form as $\frac{B}{2} \sum_{a=1}^{N} (Y_{a1} F_{a2})$.

In the end of this section, we want to note that it would be interesting to push further the similarity between the fluid of D-strings and the usual FQH systems. As a next step, it is important to build the ground state $|\Phi_0 >$ of the quantized D-string model which may be done by extending the matrix model approach of Susskind and Polychronakos. Recall in passing that the fundamental wave function of standard FQH system on plane with filling fraction $\nu = \frac{1}{k}$ is described by the Laughlin wave,

$$\Phi_L (x_1, ..., x_N) \sim \prod_{a < b=1}^{N} (x_a - x_b)^k e^{-B \sum_{a=1}^{N} |x_a|^2}. \quad (4.36)$$

This wave function, which has been conjectured long time ago by Laughlin has been recently rederived rigorously in [16] by using matrix model method. Notice that under permutation of particles, the wave function behaves as,

$$\Phi_L (x_1, ..., x_a, ..., x_b, ..., x_N) = (-)^k \Phi_L (x_1, ..., x_b, ..., x_a, ..., x_N). \quad (4.37)$$

Symmetry property of this function requires that $k$ should be a positive odd integer for a system of fermions and an even integer for bosons.
5 Conclusion and outlook

In this paper, we have developed a gauge field theoretical model proposal for a classical fluid of D-strings running in conifold and made comments on its quantum behaviour. The field action $S_{DSF}$ of this classical conifold model, in presence of a strong and constant RR background magnetic field $B$, exhibits a set of remarkable features. It is a complex holomorphic functional $S_{DSF} [\mathcal{X}, \mathcal{Y}, C_0, *] = \int_{T^\ast S^3} d^3v \mathcal{L}_{DSF} (\mathcal{X}, \mathcal{Y}, C_0, *)$ with $\mathcal{L}_{DSF} = \mathcal{L}_{DSF} (\mathcal{X}, \mathcal{Y}, C_0, *)$ given by,

$$L_{DSF} = i \frac{B_{RR}}{2\mu} (\mathcal{Y}_i \partial_0 \mathcal{X}^i - \mathcal{X}^i \partial_0 \mathcal{Y}_i) + \frac{B_{RR}}{2\mu} * (\mathcal{Y}_i \mathcal{X}^i - \mu) - \frac{B_{RR}}{\mu} C_0 (D_{++} \mathcal{Y}_i D_{--} \mathcal{X}^i - D_{--} \mathcal{Y}_i D_{++} \mathcal{X}^i),$$  \hspace{1cm} (5.1)

where $*$ is a Lagrange gauge field capturing the conifold hypersurface. By setting $\{F, G\}_{+-} = (D_{++} F) (D_{--} G) - (D_{--} F) (D_{++} G)$ and using general properties of the Poisson bracket, in particular antisymmetry and Jacobi identity as well as the property,

$$C_0 \{\mathcal{Y}_i, \mathcal{X}^i\}_{+-} = -\mathcal{Y}_i \{C_0, \mathcal{X}^i\}_{+-} - C_0 \mathcal{Y}_i \mathcal{X}^i + (D_{++} J_{--} + D_{--} J_{++}),$$ \hspace{1cm} (5.2)

with $J_{\pm\pm} = \pm (C_0 \mathcal{Y}_i D_{\pm\pm} \mathcal{X}^i)$, the above holomorphic Lagrangian density $\mathcal{L}_{DSF}$ can be also put into a gauge covariant way as follows,

$$\mathcal{L}_{DSF} = i \frac{B_{RR}}{2\mu} (\mathcal{Y}_i D_0 \mathcal{X}^i - \mathcal{X}^i D_0 \mathcal{Y}_i) + \frac{B_{RR}}{2\mu} \Lambda (\mathcal{Y}_i \mathcal{X}^i - \mu),$$ \hspace{1cm} (5.3)

with $D_0 \mathcal{X}^i = \partial_0 \mathcal{X}^i + i \{C_0, \mathcal{X}^i\}_{+-}$. The presence of the Poisson bracket $\{C_0, *\}_{+-}$ in the gauge covariant derivative $D_0$ is a signal of non commutative gauge theory in the same spirit as in Susskind description of Laughlin fluid. The basic difference is that instead of a $U(1)$ gauge group, we have here a holomorphic $\mathbb{C}^\ast$ gauge symmetry acting on scalar field as $\delta \oplus = \{\lambda, \oplus\}_{+-}$ and $\delta C_0 = \partial_0 \lambda + i \{C_0, \lambda\}_{+-}$ with $\lambda$ being the gauge parameter. Moreover, thinking about the D-string field variables as

$$\mathcal{X}^i = x^i + \mu \mathcal{C}^i_\pm,$$
$$\mathcal{Y}_i = y_i - \mu \mathcal{C}_{-i},$$ \hspace{1cm} (5.4)

where the gauge fields $\mathcal{C}^i_\pm$ describe fluctuations around the static solution, $\mathcal{L}_{DSF}$ can be put in the form (3.32) defining a complex holomorphic extension of the usual non commutative Chern-Simons gauge theory. Notice that the role of the non commutative parameter $\theta$ of usual FQH liquid is now played by the complex modulus $\mu$ of the conifold in agreement with the observation of [9]. The topological gauge theory derived in this paper may be then thought of as enveloping Susskind description of fractional quantum Hall fluid in Laughlin state. The latter follows by restricting the conifold analysis to its
Lagrangian sub-manifolds by using eqs(2.3). From this view the D-strings fluid constitutes a unified description of systems of FQH fluids in real three dimensions, in particular those involving $R \times S^2$ and $S^3$ geometries recovered as real slices of the conifold. The first geometry is obtained by restricting world sheet variable $\xi = t + i\sigma$ to its real part and the second geometry is recovered by identifying $\xi$ with $\sigma$; that is a periodic time. For instance, the restriction of eq(5.1) to the real three sphere reads as

$$
\mathcal{L}_{\text{FQH}}^{\text{real}} = \frac{\text{Re} (B_{RR})}{2 \text{Re} (\mu)} \left[ i \left( \bar{X}^i \partial_0 X^i - X^i \partial_0 \bar{X}^i \right) - 2 C_0 D_{++} \bar{X}^i D_{--} X^i \right] + \frac{\text{Re} (B_{RR})}{2 \text{Re} \mu} \left[ 2 C_0 D_{--} \bar{X}^i D_{++} X^i + * \left( \bar{X}^i X^i - \text{Re} \mu \right) \right],
$$

where $X^i = X^i (\sigma, x, \bar{x})$, $\bar{X}^i = \bar{X}^i (X^i)$, $C_0 = \bar{C}_0$, $* = \bar{*}$ and

$$
D_{++} = \frac{x^i \partial}{\partial x^i}, \quad D_{--} = \frac{\bar{x}^i \partial}{\partial \bar{x}^i}, \quad D_0 = [D_{++}, D_{--}].
$$

This analysis may be also viewed as a link between, on one hand, topological strings on conifold, and, on the other hand, non commutative Chern Simons gauge theory as well as FQH systems in real three dimensions. It would be interesting to deeper this relation which may be used to approach attractor mechanism on flux compactification by borrowing FQH ideas. To that purpose, one should first identify the matrix model regularization of the continuous field theory developed in this paper. This may be done by extending the results of [13, 14] obtained in the framework of fractional quantum Hall droplets. An attempt using matrix field variables valued in $GL (N, \mathbb{C})$ representations is under study in [15], progress in this direction will be reported elsewhere.

Acknowledgement

This research work is supported by the program Protars III D12/25, CNRST.

References


[9] EL Hassan Saidi, Topological SL(2) gauge theory on conifold and non commutative geometry, Lab/UFR-HEP/0514, GNPHE/0514, VACBT/0514


[14] EL Hassan Saidi, Topological matrix model proposal for Laughlin wave and cousin state, Lab/UFR-HEP0517/GNPHE/0519/VACBT/0519

