LATTICE DESIGN IN HIGH-ENERGY PARTICLE ACCELERATORS

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Abstract
This lecture introduces storage-ring lattice design. Applying the formalism that has been established in transverse beam optics, the basic principles of the development of a magnet lattice are explained and the characteristics of the resulting magnet structure are discussed. The periodic assembly of a storage ring cell with its boundary conditions concerning stability and scaling of the beam optics parameters is addressed as well as special lattice structures: drifts, mini beta insertions, dispersion suppressors, etc. In addition to the exact calculations indispensable for a rigorous treatment of the matter, scaling rules are shown and simple rules of thumb are included that enable the lattice designer to do the first estimates and get the basic numbers ‘on the back of an envelope’.

1 INTRODUCTION
The study of nuclear and high-energy physics has always been a driving force for the development of high energetic particle beams; while the first experiments in that field were performed using ‘beams’ from natural radioactive particle sources like α- or β- emitters, it soon became clear that to get higher energetic particles, special machines — particle accelerators — had to be developed.

One of the very first and most important experiments in twentieth century physics — the Rutherford scattering experiment in 1911 — was performed using alpha particles from a natural source in the MeV range (Fig. 1, [1]), before particle accelerators had been invented.

Fig. 1: Result of the Rutherford scattering experiment compared to the angle expected from the Thomson model
Soon afterwards, machines to accelerate particle beams artificially, such as electrostatic machines like the Cockroft–Walton generator and the Van de Graaf accelerator as well as the first circular accelerators like the betatron and the cyclotron [2] came into operation.

The basis for a concept called ‘lattice design’, the topic of this lecture, was laid out in 1952 when Courant, Livingston, and Snyder developed the theory of strong focusing accelerators (or alternating-gradient machines) [3].

Lattice design in the context described here is the design and optimization of the principal elements — the lattice cells — of a (circular) accelerator, and it includes the dedicated variation of lattice elements (for example, position and strength of the magnets in the machine) to get well-defined and predictable parameters of the stored particle beam.

It is therefore closely related to the theory of linear beam optics described in Ted Wilson’s lecture in this school [4] and in the introductory CAS lecture by Rossbach and Schmueser [5].

Machine lattice design is an application of linear beam optics and has high practical relevance. Lattice examples of present-day storage rings and plots from optics calculations of real machines are shown throughout the paper.

In Fig. 2 the beam optics in a part of a lattice structure is shown. The plot shows the most important beam parameters in a typical high-energy storage ring. In the upper part, the beta function in the horizontal ($x$) and vertical ($y$) plane is plotted (or the square root proportional to the beam size), in the middle part the position of the lattice elements is shown, and the lower part shows the so-called dispersion function in $x$ and $y$. We will discuss these parameters in some detail in this paper.

![Fig. 2: Lattice and beam optics in a part of a typical high-energy accelerator. The curves in the upper part refer to the square root of the beta function, the lower part shows the dispersion function.](image)

### 1.1 Geometry of the ring

The first step in the layout of a storage ring is fixing the geometry or, the closely related task, determining the momentum of the particles to be stored in it.
For the bending force and focusing of the particle beam, magnetic fields are applied in a circular accelerator. In principle, electrostatic fields would be possible as well but at high momenta (i.e. if the particle velocity is close to the speed of light) magnetic fields are much more efficient. The force acting on the particles, the Lorentz force, is given by
\[ F = q \times E + (v \times B) \]
or neglecting the \( E \) field,
\[ F = q \times (v \times B) . \]

In a constant transverse magnetic field \( \vec{B} \), the particle will see a constant deflecting force and the trajectory will be a part of a circle. In other words, the condition for a circular orbit is that the Lorentz force be equal to the centrifugal force:
\[ e* \nu \* B = \frac{mv^2}{\rho} . \]

Dividing by the velocity \( \nu \) we get a relation between the magnetic field and the momentum of the particle
\[ e* B = \frac{mv}{\rho} = \frac{p}{\rho} . \]
The term \( B*\rho \) is called beam rigidity,
\[ B* \rho = \frac{p}{e} \]
and connects the magnetic dipole field needed for a circular orbit of radius \( \rho \) to the particles’ momentum and charge. (Note that here we often refer to protons or electrons and the charge is just the elementary charge \( e \)). With an ideal circular orbit, for each segment of the path we get the relation
\[ \alpha = \frac{ds}{\rho}, \quad \alpha = \frac{B*ds}{B* \rho} \]
and integrating along the path \( ds \) in all dipole magnets in the ring we require
\[ \alpha = \frac{\int Bds}{B* \rho} = 2\pi \rightarrow \int Bds = 2\pi \frac{p}{q} . \quad (1) \]

As in any ‘circular’ accelerator the angle swept in one turn for the design particle is \( 2\pi \). Equation (1) tells us that the integral of all bending magnets in the ring has to be \( 2\pi \) times the momentum of the beam. If the path length inside the dipole magnet does not differ much from the length of the magnet itself, the integral in Eq. (1) can be approximated by \( \int Bdl \), where \( dl \) refers to the magnet length.

In Fig. 3 the field of a typical bending magnet used in a storage ring is shown. On the vertical scale, the magnetic induction \( B \) is shown in tesla, and is measured between the two pole faces of the dipole magnet. The plateau of constant field is easily seen inside the magnet as are the decreasing edge fields before and after the magnet.
For the lattice designer, the integrated $B$ field along the particles’ design orbit (roughly sketched in the figure) is the most important parameter, as it is the value that enters Eq. (1) and defines the field strength and how many of these magnets are needed for a full circle.

Figure 4 shows a photograph of a small storage ring [6] with only eight dipole magnets used to define the design orbit. The magnets are powered symmetrically and therefore each magnet corresponds to a bending angle of exactly $\alpha = 45^\circ$ of the beam. The field strength in this machine is in the order of $B = 1$ T.

In general, for a high-energy storage ring or synchrotron, a large number of bending magnets with very high magnetic fields are needed to determine the design orbit.

As an example, the HERA storage ring is presented in Fig. 5. It accelerates and stores proton beams of an energy of 920 GeV and collides them with $e^+$ or $e^-$ beams of about 27.5 GeV. In the HERA proton ring, 416 dipole magnets are used to guide the beam on an orbit of 6.3 km circumference. The length of each dipole magnet is $l = 8.8$ m.

At an energy of 920 GeV the particles are ultra-relativistic and we can put

$$E \approx p \times c$$
to calculate the momentum
\[ \int Bdl = N * l * B = 2\pi p / q \] (2)

Using formula (2), for the magnetic field we get
\[ B \approx \frac{2\pi * 920 * 10^9 \text{eV}}{416 * 3 * 10^8 \text{m/s} * 8.8 * 10^3 \text{m} / \text{e}} \approx 5.15 \text{T} \]

It is immediately clear that the machine has to be built with superconducting magnets if the required energy of 920 GeV is to be achieved.

Fig. 5: HERA storage ring: 416 superconducting magnets are needed to bend the protons on a circular path of 6.3 km length

1.2 Equation of motion and matrix formalism

If the geometry and specification of the arc is determined and the layout of the bending magnets is accomplished, the next step is the focusing properties of the machine. In general, we have to keep more than \(10^{12}\) particles in the machine, distributed in a certain number of bunches. And these particles have to be focused to keep their trajectories close to the design orbit.

As mentioned in the linear beam optics lecture, gradient fields are used to do the job. They generate a magnetic field that increases linearly as a function of the distance to the magnet centre;

\[ B_x = -g * x, \quad B_y = -g * y \]

where \(x\) and \(y\) refer to the horizontal and vertical plane and the parameter \(g\) is called the gradient of the magnetic field.

It is the custom to normalize the magnetic fields to the momentum of the particles. In the case of the dipole fields we get from Eq. (1)

\[ \alpha = \frac{\int Bdl}{B * \rho} = \frac{L_{\text{eff}}}{\rho} \]
where \( L_{\text{eff}} \) is the so-called effective length of the magnet; the term \( 1/\rho \) is the bending strength of the dipole. In the same way the field of the quadrupole lenses is normalized to \( B^*\rho \): the strength \( k \) is defined by

\[
  k = \frac{g}{B^*\rho}
\]

and the focal length of the quadrupole is given by

\[
  f = \frac{1}{k^*\ell}.
\]

Under the influence of the focusing properties of the quadrupole and dipole fields in the ring, the particle trajectories are described by a differential equation. In the lecture about linear beam optics this equation is derived in all its glory, so here we just state that it is given by the expression

\[
  x'' + K^*x = 0
\]

\( x \) describes the horizontal coordinate of the particle with respect to the design orbit, the derivative in linear beam optics is taken with respect to the orbit coordinate \( s \), and the parameter \( K \) combines the focusing strength \( k \) of the quadrupole and the weak focusing term \( 1/\rho^2 \) of the dipole field. (Note: A negative value of \( k \) means a horizontal focusing magnet.)

\[
  K = -k + 1/\rho^2.
\]

In most accelerators, the term \( 1/\rho^2 \) is missing in the vertical plane, the design orbit is in the horizontal plane and no vertical bending strength is present. So we get in the vertical plane

\[
  K = k.
\]

Remember when designing a magnet lattice to simplify as much as possible in the beginning. Clearly the exact solution of particle motion has to be calculated in full detail and if the beam optics is optimized on a linear basis, higher-order multipole fields and their effect on the beam have to be taken into account. By completing the first steps we can make life easier and ignore terms small enough to be neglected.

In many cases the weak focusing term \( 1/\rho^2 \) can be neglected in favour of a rough estimate making the formula much shorter and symmetric in the horizontal and vertical plane. Referring to the example of the HERA proton ring, the basic parameters of the machine are as in Table 1.

| Table 1: Basic parameters of the HERA proton storage ring |
|---------------------------------|------------------|
| Circumference \( C_0 \)          | 6335 m           |
| Bending radius \( \rho \)        | 580 m            |
| Quadrupole gradient \( G \)      | 110 T/m          |
| Particle momentum \( p \)        | 920 GeV/c        |
| Weak focusing term \( 1/\rho^2 \) | 2.97*10^{-6} 1/m^2 |
| Focusing strength \( k \)        | 3.3*10^{-3} 1/m^2 |

The weak focusing contribution \( 1/\rho^2 = 2.97 *10^{-6} /m \) is much smaller than the quadrupole strength \( k \), and can generally be neglected for initial estimates in the lattice of large accelerators.
1.3 Single-particle trajectories

The differential Eq. (3) describes the transverse motion of the particle with respect to the design orbit. In linear approximation it can be solved and the solutions for the horizontal and vertical planes are independent of each other.

If the focusing parameter $K$ is constant, if we refer the situation to a place inside a magnet where the field is constant along the orbit, the general solution for the position and angle of the trajectory can be derived as a function of the initial conditions $y_0$ and $y'_0$. In the case of a focusing lens we obtain

$$x(s) = x_0 \cos(\sqrt{K} s) + \frac{x'_0}{\sqrt{K}} \sin(\sqrt{K} s)$$

$$x'(s) = -x_0 \sqrt{K} \sin(\sqrt{K} s) + x'_0 \cos(\sqrt{K} s)$$

Or, written in a more convenient matrix form:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = M \begin{pmatrix} x \\ x'_0 \end{pmatrix}$$

The matrix $M$ depends on the properties of the magnet, and for a number of typical lattice elements we get

focusing quadrupole

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} s) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} s) \\ -\sqrt{K} \sin(\sqrt{K} s) & \cos(\sqrt{K} s) \end{pmatrix}$$  \hspace{1cm} (4a)$$

defocusing quadrupole

$$M_{QD} = \begin{pmatrix} \cosh(\sqrt{K} s) & \frac{1}{\sqrt{K}} \sinh(\sqrt{K} s) \\ \sqrt{K} \sinh(\sqrt{K} s) & \cosh(\sqrt{K} s) \end{pmatrix}$$  \hspace{1cm} (4b)$$

drift space

$$M_{drift} = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$  \hspace{1cm} (4c)$$

1.4 The Twiss parameters $\alpha, \beta, \gamma$

In the case of periodic conditions in the accelerator there is another more convenient way to describe the particle trajectories. The above-mentioned formalism is valid only within a single element.

Note that in a circular accelerator the focusing elements are periodic in the orbit coordinate $s$ after one revolution. In addition, storage ring lattices have, in most cases, an inner periodicity. They often are built — at least partly — of sequences where identical magnetic cells, the lattice cells, are repeated several times in the ring and lead to periodically repeated focusing properties.

In this case the transfer matrix from the beginning of the structure to the end is expressed as a function of the periodic parameters $\alpha, \beta, \gamma, \phi$: 
\[
M(s) = \begin{pmatrix}
\cos(\varphi) + \alpha_s \sin(\varphi) & \beta \sin(\varphi) \\
-\gamma_s \sin(\varphi) & \cos(\varphi) - \alpha_s \sin(\varphi)
\end{pmatrix}.
\]  

(5)

The parameters \(\alpha\) and \(\gamma\) are related to the \(\beta\)-function by the equations

\[
\alpha(s) = -\frac{1}{2} \beta'(s) \quad \text{and} \quad \gamma(s) = \frac{1 + \alpha^2(s)}{\beta(s)}.
\]

The matrix is clearly a function of the position \(s\), as the parameters \(\alpha, \beta, \gamma\) depend on \(s\). The variable \(\varphi\) is called phase advance of the trajectory and is given by

\[
\varphi = \frac{s + L}{\beta(t)} dt.
\]

In such a periodic lattice, for stability of the equation of motion the relation

\[|\text{trace}(M)| < 2\]

setting boundary conditions for the focusing properties of the lattice, has to be valid.

Given that correlation, the solution of the trajectory of a particle can be expressed as a function of these new parameters:

\[
y(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cos(\varphi(s) - \delta)
\]

\[
y'(s) = \frac{-\sqrt{\varepsilon} \sqrt{\beta(s)}}{\sqrt{\beta(s)}} \left\{ \sin(\varphi(s) - \delta) + \alpha(s) \cos(\varphi(s) - \delta) \right\}.
\]

The position and angle of the transverse oscillation of a particle at a point \(s\) is given by the value of the \(\beta\)-function at that location and \(\varepsilon\) and \(\delta\) are constants of the particular trajectory.

Finally, in terms of linear beam optics, the Twiss parameters at a position \(s\) in the lattice are defined by the focusing properties of the complete storage ring. They are transformed through the lattice from one point to another by the matrix elements of the corresponding magnets. Without proof, if matrix \(M\) is given by

\[
M(s_1,s_2) = \begin{pmatrix}
C & S \\
C' & S'
\end{pmatrix}
\]

then the transformation rule from point \(s_1\) to \(s_2\) in the lattice is given by

\[
\begin{pmatrix}
\beta \\
\alpha \\
\gamma
\end{pmatrix}_{s_2} = \begin{pmatrix}
C^2 & -2SC & S^2 \\
-CC' & SC' + S'C & -SS' \\
C'^2 & -2S'C' & S'^2
\end{pmatrix} \begin{pmatrix}
\beta \\
\alpha \\
\gamma
\end{pmatrix}_{s_1}.
\]

(6)

The terms \(C, S\) etc. correspond to the focusing properties of the matrix. For one single element, see the expressions given in the Eq. (4) of the previous section.
2 LATTICE DESIGN

A simple drift space in a lattice is the easiest case to investigate.

In Eq. (4) the matrix for a drift space is given by

\[
M \left( \begin{array}{cc} C & S \\ C' & S' \end{array} \right) = \left( \begin{array}{cc} 1 & \ell \\ 0 & 1 \end{array} \right).
\]

(7)

Starting with position \(x_0\) and angle \(x_0'\) the trajectory after the drift therefore will be

\[
\begin{pmatrix} x \\ x' \end{pmatrix}_\ell = \left( \begin{array}{cc} 1 & \ell \\ 0 & 1 \end{array} \right) \ast \begin{pmatrix} x \\ x' \end{pmatrix}_0
\]

or written explicitly:

\[
x(\ell) = x_0 + \ell \ast x_0
\]

\[
x'(\ell) = x_0'.
\]

If the drift is located in a circular accelerator or within a periodic part of a lattice, the Twiss parameters are well-defined at the start of the drift, and they will be transformed according to Eq. (6) from their initial values \(\alpha_0, \beta_0, \gamma_0\) via

\[
\begin{bmatrix} \beta \\ \alpha \\ \gamma \end{bmatrix}_\ell = \left( \begin{array}{ccc} 1 & -2\ell & \ell^2 \\ 0 & 1 & -\ell \ast \beta \\ 0 & 0 & 1 \end{array} \right) \ast \begin{bmatrix} \beta \\ \alpha \\ \gamma \end{bmatrix}_0
\]

The beta function in the drift develops as

\[
\beta(\ell) = \beta_0 - 2\ell \ast \alpha_0 + \ell^2 \ast \gamma_0.
\]

Calculating the trace of the matrix (7) we see that the condition for stability is unfulfilled.

\[
|trace(M_\ell)| = 1 + 1 = 2.
\]

So we know that a circular accelerator built exclusively out of drift spaces is not a stable machine. (This is a pity, since it would have been a cheap machine.)

N.B., in any storage ring there will be a large number of drift spaces between the focusing elements. The stability criterion requires the magnetic elements and the drift spaces in between them to be arranged so that the resulting lattice cell describes a stable solution in both the horizontal and vertical plane.

Figure 6 shows the lattice and the beam optics of a typical high-energy storage ring, the HERA proton ring already mentioned. In the upper part of Fig. 6, the square root of the \(\beta\) function is shown in both transverse planes. The broad band in the middle is the lattice. As the ring has a circumference of about 6.3 km the single lattice elements can not be clearly distinguished in the figure. In the lower part, the dispersion function is shown.
Initially, two sections of quite different characteristics can be identified in the accelerator and, in fact, most if not all, high-energy colliders are designed with:

- A section where the beta function shows a regular pattern: These are the arcs where the main bending magnets of the ring are located. They define the geometry of the ring and the maximum magnetic fields of the bends limit the energy (or momentum) of the particle beam. In addition, the main focusing elements of the ring are located in the arcs where quadrupole lenses for tune control and sextupole magnets for the compensation of the chromaticity of the optics are usually found.

- Regular arcs connected by straight sections in the ring: These are long lattice parts where the optics is modified to establish conditions needed for particle injection, to reduce the dispersion function, or where the beam dimensions are reduced to increase the particle collision rate in the case of a collider ring. Beyond that, all kind of devices have to be installed in these long sections like RF cavities, beam diagnostic tools, and even high-energy particle detectors (…if they cannot be avoided in the machine).

Concerning the arcs’ structure it is advantageous to configure them on the basis of small elements — called cells — repeated many times in the ring. One of the most widespread lattice cells used for this purpose is the FODO cell.

### 2.1 The FODO cell

A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with basically nothing in between, is called a FODO lattice, and the elementary cell a FODO cell. ‘Basically nothing’ in that context means any element with negligible effect on the focusing properties, as, for example, drift spaces, RF-structures or, under certain circumstances, even bending magnets. A FODO cell is shown in Fig 7.
To calculate the Twiss parameters $\alpha$, $\beta$, $\gamma$ of a FODO cell, we start our calculations in the middle of the focusing quadrupole. And to start with the most simple configuration, the drift spaces between the two quadrupole magnets will really be empty and of equal length.

In Fig. 8 the optical solution of a typical arc structure is shown. Many FODO cells are connected with each other and the optical functions are calculated with a beam optics program. The plot shows the $\beta$ function in both planes and below the position of the magnet lenses, the lattice. The solid line shows the horizontal beta function, the dashed one the vertical $\beta$. Obviously, the solution for both $\beta$’s is periodic.

Can we understand what the optics code is doing? To answer this question we refer to a single cell: Qualitatively Fig. 8 clearly illustrates that the horizontal $\beta_x$ reaches its maximum value at the centre of the (horizontal) focusing quadrupoles and its minimum value at the defocusing lenses and vice versa for the vertical function $\beta_y$. Table 2 shows the numerical result of the optics code.

<table>
<thead>
<tr>
<th>Element</th>
<th>$l$ (m)</th>
<th>$k$ (1/m²)</th>
<th>$\beta_x$ (m)</th>
<th>$\alpha_x$</th>
<th>$\phi_x$ (rad)</th>
<th>$\beta_y$ (m)</th>
<th>$\alpha_y$</th>
<th>$\phi_y$ (rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Start</td>
<td>0</td>
<td>–</td>
<td>11.611</td>
<td>0</td>
<td>0</td>
<td>5.295</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>QFH</td>
<td>0.25</td>
<td>-0.0541</td>
<td>11.228</td>
<td>1.514</td>
<td>0.0110</td>
<td>5.488</td>
<td>-0.78</td>
<td>0.0220</td>
</tr>
<tr>
<td>QD</td>
<td>3.251</td>
<td>0.0541</td>
<td>5.4883</td>
<td>-0.78</td>
<td>0.2196</td>
<td>11.23</td>
<td>1.514</td>
<td>0.2073</td>
</tr>
<tr>
<td>QFH</td>
<td>6.002</td>
<td>-0.0541</td>
<td>11.611</td>
<td>0</td>
<td>0.3927</td>
<td>5.295</td>
<td>0</td>
<td>0.3927</td>
</tr>
<tr>
<td>End</td>
<td>6.002</td>
<td>–</td>
<td>11.611</td>
<td>0</td>
<td>0.3927</td>
<td>5.295</td>
<td>0</td>
<td>0.3927</td>
</tr>
</tbody>
</table>
The $\alpha$-function in the middle of the quadrupole is zero and, as $\alpha(s) = -\beta'(s)/2$, the $\beta$ function is maximum or minimum at that position.

We know from the linear optics lecture that the phase advance of the complete machine is called the working point, and it is counted in units of $2\pi$. In our case we chose $\varphi = 45^\circ$ or 0.3927 rad as phase advance of one single cell and the corresponding working point would be

$$Q_c = \frac{\varphi ds}{2\pi} = 0.125.$$ 

Lastly, as we have chosen equal quadrupole strengths in both planes, i.e. $k_x = -k_y$ and uniform drift spaces between the quadrupoles, the lattice is called a symmetric FODO cell. And therefore we expect symmetric optical solutions in the two transverse planes.

In linear beam-optics the transfer matrix of a number of optical elements is given by the product of the matrices of the single elements introduced in Eq. (4). In our case we get

$$M_{FODO} = M_{QFH} \ast M_{LD} \ast M_{QD} \ast M_{QFH}.$$  \hspace{1cm} (8)

However, as we have decided to start the calculation in the middle of a quadrupole magnet, the corresponding matrix has to take that into account. The first matrix will be that of a half-quadrupole. Putting in the numbers for the length and strength, $k = \pm 0.54102/m^2$, $l_q = 0.5m$, $l_d = 2.5 m$ we get

$$M_{FODO} = \begin{pmatrix} 0.707 & 8.206 \\ -0.061 & 0.707 \end{pmatrix}.$$ 

This matrix describes uniquely the optical property of the lattice and defines the beam parameters.

2.1.1 The most important point: stability of the motion

The trace of $M$ gives

$$|\text{trace}(M_{FODO})| = 1.415 < 2.$$ 

A lattice built out of these FODO cells provides stable conditions for the particle motion. However, if we introduce new parts in the lattice we have to go through the calculation again.

In addition the matrix can be used to determine the optical parameters of the system:

2.1.2 Phase advance per cell

Writing $M$ as a function of $\alpha$, $\beta$, $\gamma$ and the phase advance $\varphi$ we get:

$$M(s) = \begin{pmatrix} \cos(\varphi) + \alpha \sin(\varphi) & \beta \sin(\varphi) \\ -\gamma \sin(\varphi) & \cos(\varphi) - \alpha \sin(\varphi) \end{pmatrix}$$  \hspace{1cm} (9)

and we immediately see that

$$\cos(\varphi) = \frac{1}{2} \times \text{trace}(M) = 0.707$$

or $\varphi = 45^\circ$, which corresponds to the working point of 0.125 calculated above.

2.1.3 The $\alpha$ and $\beta$ functions are calculated in a similar way

For $\beta$ we use the relation
\[ \beta = \frac{M(1,2)}{\sin(\varphi)} = 11.611 \text{ m} \]

and we obtain \( \alpha \) through the expression
\[ \alpha = \frac{M(1,1) - \cos(\varphi)}{\sin(\varphi)} = 0. \]

To complete this look at the optical properties of a lattice cell, I want to give a rule of thumb for the working point: Defining an average \( \beta \) function of the ring we put
\[ \int \frac{ds}{\beta} = \frac{L}{\bar{\beta}}. \]

With \( L = 2\pi \bar{R} \) and \( \bar{R} \) being the average bending radius of the ring (which is not the bending radius of the dipole magnets) for the working point \( Q \) we can write
\[ Q = N^* \frac{\phi_c}{2\pi} = \frac{1}{2\pi} \int \frac{ds}{\beta(s)} = \frac{1}{2\pi} \frac{2\pi \bar{R}}{\bar{\beta}}, \]

where \( N \) is the number of cells and \( \phi_c \) denotes the phase advance per cell. So we get
\[ Q \approx \frac{\bar{R}}{\bar{\beta}}. \]

A rough estimate for the working point is obtained by the ratio of the mean radius of the ring and the average \( \beta \) function of the lattice.

2.2 Thin-lens approximation

As discussed, a first estimate of the parameters of a lattice can and should be carried out at the beginning of a magnet lattice design.

If we want fast answers and require only rough estimates we can simplify. Under certain circumstances the matrix of a focusing element can be written in the thin-lens approximation. Given, for example, the matrix of a focusing lens
\[ M_{QF} = \begin{pmatrix} \cos(\sqrt{K} \ell) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} \ell) \\ -\sqrt{K} \sin(\sqrt{K} \ell) & \cos(\sqrt{K} \ell) \end{pmatrix}, \]

we can simplify the trigonometric terms if the focal length of the quadrupole magnet is much larger than the length of the lens:
\[ f = \frac{1}{k\ell_Q} \gg \ell_Q, \]

the transfer matrix can be approximated using \( kl_Q = \text{const}, l_Q \to 0 \) and we get
\[ M_{QF} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}. \]
Referring to the notation used in Fig. 7 we can calculate the transfer matrix first from the middle of the focusing to the middle of the defocusing quadrupole and get the matrix for half the cell:

\[
M_{\text{halfCell}} = M_{QD/2} M_{\ell_D} M_{OF/2}
\]

\[
M_{\text{halfCell}} = \begin{pmatrix}
\frac{1}{f} & 0 \\
\ell_D & 1
\end{pmatrix} \begin{pmatrix}
1 & \ell_D \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
-\ell_D & 1
\end{pmatrix}
\]

\[
M_{\text{halfCell}} = \begin{pmatrix}
1 - \frac{\ell_D}{f} & \ell_D \\
-\ell_D & 1 + \frac{\ell_D}{f}
\end{pmatrix} \quad \text{(10)}
\]

Note that the thin-lens approximation implies that \( l_0 \to 0 \), therefore the drift between the magnets has to be \( \ell_D = L/2 \). As we are now dealing with half-quadrupoles, for the focal length of a half-quadrupole we have set \( \tilde{f} = 2f \). We get the second half of the cell by replacing \( \tilde{f} \) by \( -\tilde{f} \), and the matrix for the complete FODO in thin-lens approximation is

\[
M = \begin{pmatrix}
1 + \frac{\ell_D}{f} & \ell_D \\
-\ell_D & 1 - \frac{\ell_D}{f}
\end{pmatrix} \begin{pmatrix}
1 - \frac{\ell_D}{f} & \ell_D \\
-\ell_D & 1 + \frac{\ell_D}{f}
\end{pmatrix}
\]

or multiplying

\[
M = \begin{pmatrix}
1 - \frac{2\ell_D^2}{f^2} & 2\ell_D(1 + \frac{\ell_D}{f}) \\
2\ell_D(1 + \frac{\ell_D}{f}) & 1 - \frac{2\ell_D^2}{f^2}
\end{pmatrix} \quad \text{(11)}
\]

The matrix is now much easier to handle than the equivalent formula (4) and (8). And the approximation is, in general, not bad.

Going briefly through the calculation of the optics parameters again, according to (9) and (11) we immediately get

\[
\cos(\varphi) = 1 - \frac{2\ell_D^2}{f^2},
\]

and with a little bit of trigonometric gymnastics

\[
1 - 2\sin^2(\varphi/2) = 1 - \frac{2\ell_D^2}{f^2}
\]

we can simplify this expression and get

\[
\sin(\varphi/2) = \ell_D / \tilde{f} = \frac{L_{\text{Cell}}}{2f}.
\]
and finally
\[ \sin(\varphi/2) = \frac{L_{\text{cell}}}{4f}. \]  

(12)

In thin-lens approximation the phase advance of a FODO cell is given by the length of the cell \( L_{\text{cell}} \) and the focal length of the quadrupole magnets \( f \).

For the parameters of the example given above, we get a phase advance per cell of \( \varphi \approx 47.8^\circ \), and in full analogy to the calculation presented previously, we calculate \( \beta \approx 11.4 \text{ m} \), which is very close to the result of the exact calculations (\( \varphi = 45^\circ, \beta = 11.6 \text{ m} \)).

2.2.1 Stability of the motion
In thin-lens approximation, the condition for stability \(|\text{trace}(M)| < 2\) requires that
\[ 2 - \frac{4f^2}{D^2} < 2 \]
or
\[ f > \frac{L_{\text{cell}}}{4}. \]  

(13)

We have the important and simple result that for stable motion the focal length of the quadrupole lenses in the FODO has to be larger than a quarter of the length of the cell.

2.3 Scaling optical parameters of a lattice cell
After the discussion on stability in a lattice cell and the first estimates and calculations of the optical functions \( \alpha, \beta, \gamma \) and \( \varphi \), we would like to concentrate a little more on a detailed analysis of a FODO with these parameters.

We can calculate the \( \beta \)-function that corresponds to the periodic solution — provided we know the strength and length of the focusing elements in the cell. But can we optimize somehow?

In other words, for a given lattice, what would be the ideal magnet strength to get the smallest beam dimensions? To answer this question, look at the transfer matrix of half a FODO cell in Eq. (10), i.e. the transfer from the middle of a QF quadrupole to the middle of a QD (see Fig. 7).

From linear beam optics we know that the transfer matrix between two points in a lattice can be expressed not only as a function of the focusing properties of the elements in that section of the ring, but in an equivalent way as a function of the optical parameters between the two reference points. For a full turn or, within a periodic lattice for one period, we have used that relation already in Eq. (5).

The general expression — in the non periodic case — reads [5]
\[ M_{1 \rightarrow 2} = \begin{pmatrix} \frac{\beta_2}{\beta_1} (\cos \Delta \varphi + \alpha \sin \Delta \varphi) & \sqrt{\beta_2 \beta_1} \sin \Delta \varphi \\ (\alpha_1 - \alpha_2) \cos \Delta \varphi - (1 + \alpha_1 \alpha_2) \sin \Delta \varphi & \sqrt{\beta_2} \beta_1 (\cos \Delta \varphi - \alpha_2 \sin \Delta \varphi) \end{pmatrix}. \]  

(14)
The indices refer to the starting point \( s_1 \) and the end point \( s_2 \) in the ring and \( \Delta \phi \) is the phase advance between these points. It is evident that this matrix is reduced to the form given in Eq. (5) if the periodic conditions \( \beta_1 = \beta_2 \), \( \alpha_1 = \alpha_2 \) are fulfilled.

In the middle of the focusing quadrupole \( \beta \) reaches its highest value and in the middle of the defocusing magnet its lowest one (referring to the vertical plane the argument is valid ‘vice versa’) and the \( \alpha \) functions at that position are zero. Therefore the transfer matrix can be written in the form

\[
M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{\beta}}{\sqrt{\beta}} & \sqrt{\beta} \sin \Delta \phi \\ -1 & \sqrt{\beta} \cos \Delta \phi \end{pmatrix}.
\]

Using this expression and putting the terms that we have developed in thin-lens approximation in Eq. (10) for the matrix elements, we get

\[
\frac{\hat{\beta}}{\beta} = \frac{S'}{C'} = \frac{1 + \ell_D/\bar{f}}{1 - \ell_D/\bar{f}} = \frac{1 + \sin \frac{\phi}{2}}{1 - \sin \frac{\phi}{2}}
\]

\[
\hat{\beta} = \frac{-S}{C'} = \bar{f}^2 = \frac{\ell_D^2}{\sin^2 \frac{\phi}{2}}
\]

where we have set \( \Delta \phi = \phi/2 \) for the phase advance of half the FODO cell. The two expressions for \( \hat{\beta} \) and \( \beta \) can be combined to calculate both parameters

\[
\hat{\beta} = \frac{(1 + \sin \frac{\phi}{2})L}{\sin \phi}, \quad \beta = \frac{(1 - \sin \frac{\phi}{2})L}{\sin \phi}.
\]  \hspace{1cm} (15)

We get the simple result that the maximum (and minimum) value of the \( \beta \)-function and therefore the maximum dimension of the beam in the cell is determined by the length \( L \) and the phase advance \( \phi \) of the complete cell.

Figure 9 shows a three-dimensional picture of a proton bunch for typical conditions in the HERA storage ring. The bunch length is about 30 cm and determined by the momentum spread and the RF potential [7]. The values of \( \hat{\beta} \) and \( \beta \), as determined by the cell characteristics, are normally 80 m and 40 m, respectively.
2.3.2 Optimization of the FODO phase advance

From Eq. (15) we see that — given the length of the FODO — the maximum $\beta$ depends only on the phase advance per cell. Therefore, we may ask whether there is an optimum phase that leads to the smallest beam dimension.

If we assume a Gaussian particle distribution in the transverse plane and denote the beam emittance by $\varepsilon$, the transverse beam dimension $\sigma$ is given by

$$\sigma = \sqrt{\varepsilon \beta}.$$ 

In a typical high-energy proton ring, $\varepsilon$ is in the order of some $10^{-9}$ m*rad (for example, the HERA proton ring at $E = 920$ GeV, $\varepsilon \approx 6 \times 10^{-9}$ m*rad) and as the typical $\beta$ functions are about $\beta \approx 40 \ldots 100$ m in the arc, the resulting beam dimension is roughly one millimetre. At the interaction point of two counter-rotating beams even beam radii in the order of a micrometre are obtained.

Figure 10 shows the result of a beam scan used to measure the transverse beam dimension.

In general both emittances are equal for a proton beam, $\varepsilon_x \approx \varepsilon_y$. A proton beam is ‘round’ even if the varying beta function along the lattice leads to beam dimensions in two transverse planes that can be quite different.

Optimizing the beam dimension in the case of a proton ring therefore means searching for a minimum of the beam radius:

$$r^2 = \varepsilon_x \beta_x + \varepsilon_y \beta_y$$
and therefore optimizing the sum of the maximum and minimum beta functions at the same time:

\[
\beta + \beta = \frac{(1 + \sin \frac{\phi}{2})L \sin \phi}{2} + \frac{(1 - \sin \frac{\phi}{2})L \sin \phi}{2}.
\] (16)

The optimum phase \(\phi\) is obtained by

\[
\frac{d}{d\phi} (\beta + \beta) = 0
\]

which gives

\[
\frac{L}{\sin^2 \phi} \cos \phi = 0 \rightarrow \phi = 90^\circ.
\]

As for the aperture requirement of the cell, a phase advance of \(\phi = 90^\circ\) is the best value in a proton ring. The plot of Fig. 11 shows the sum of the two \(\beta\)'s (Eq. 16) as a function of the phase \(\phi\) in the range of \(\phi = 0 \ldots 180^\circ\).

**Fig. 11:** Sum of the horizontal and vertical \(\beta\) function as a function on the phase advance \(\phi\)

Optimization of the beam radii can be a critical issue in accelerator design. Large beam dimensions need big apertures of the quadrupole and dipole magnets in the ring. Running the machine at the highest energy can lead to limitations in the focusing power as the gradient of a quadrupole lens scales as the inverse of its squared aperture radius \(r_0^2\) and increases the cost for the magnet lenses. Therefore it is recommended not to tune the lattice too far away from the ideal phase advance.

Here, for completeness, I have to make a short remark on electron machines. Unlike the situation in proton rings, electron beams are flat in general: On account of synchrotron radiation [8], the vertical emittance of an electron or positron beam is only a small fraction of the horizontal one \(\varepsilon_y \approx 1\ldots10 \% \varepsilon_x\). For the optimization of the phase advance, the calculation can and should be restricted to the horizontal plane only and the condition for smallest beam dimension is

\[
\frac{d}{d\phi} (\hat{\beta}) = \frac{d}{d\phi} \frac{L(1 + \sin \frac{\phi}{2})}{\sin \phi} = 0 \rightarrow \phi \approx 76^\circ.
\]

Figure 12 shows the horizontal and vertical \(\beta\) as a function of \(\phi\) in that case. In an electron ring the typical phase advance is in the range of \(\phi \approx 30 \ldots 90^\circ\).
2.4 Dispersion in a FODO lattice

In the design and description of the optical parameters of a magnet lattice, we have discussed the scenario of particles with ideal momentum, and as such, have not deviated from standard information presented in the literature on linear beam dynamics. In general, the energy (or momentum) of the particles stored in a ring deviate from the ideal momentum $p_0$ of the beam.

In linear beam-optics, the differential equation for the transverse movement gets an additional term if the momentum deviation is not zero: $\Delta p/p \neq 0$. This gives an inhomogeneous equation of motion

$$x'' + K(s)x = \frac{1}{\rho} \frac{\Delta p}{p}.$$  \hspace{1cm} (17)

The left-hand side of (17) is the same as in the homogeneous Eq. (3) and the parameter $K$ describes the focusing strength of the lattice element at position $s$ in the ring. As usual, the general solution of Eq. (17) is the sum of the complete solution $x_H$ of the homogeneous equation and a special solution of the inhomogeneous one $x_i$.

$$x_H'' + K(s)x_H = 0$$
$$x_i'' + K(s)x_i = \frac{1}{\rho} \frac{\Delta p}{p_0}.$$  \hspace{1cm} (18)

The special solution $x_i$ can be normalized to the momentum error $\Delta p/p$ giving the dispersion function $D(s)$

$$x_i(s) = D(s) \frac{\Delta p}{p}.$$  \hspace{1cm} (18)

Starting from the initial conditions $x_0$ and $x'_0$, the general solution for the particle trajectory now reads

$$x(s) = C(s)x_0 + S(s)x'_0 + D(s)\frac{\Delta p}{p}$$

or, including the expression for the angle $x'(s)$,

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}.$$
For convenience, extend the matrix to include the second term and write:

\[
\begin{pmatrix}
    x \\
    x' \\
    \Delta p/p \bigg|_S
\end{pmatrix} =
\begin{pmatrix}
    C & S & D \\
    C' & S' & D' \\
    0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
    x \\
    x' \\
    \Delta p/p \bigg|_0
\end{pmatrix}.
\]

The dispersion function \( D(s) \) is (obviously) defined by the focusing properties of the lattice and the bending strength of the dipole-magnets \( 1/\rho \) and it can be shown that [5]

\[
D(s) = S(s) \star \int_{\sigma_0}^{\sigma} \frac{1}{\rho(t)} C(\tilde{s}) d\tilde{s} - C(s) \star \int_{\sigma_0}^{S(s)} \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}.
\]

(19)

The variable \( s \) refers to the position where the dispersion is obtained (or measured if you like) and the integration has to be performed over all places \( \tilde{s} \) where a non-vanishing term \( 1/\rho \) exists (in general in the dipole magnets of the ring).

2.4.1 Example

The 2 × 2 matrix for a drift space is given by

\[
M_{\text{Drift}} = \begin{pmatrix}
    C & S \\
    C' & S'
\end{pmatrix} = \begin{pmatrix}
    1 & \ell \\
    0 & 1
\end{pmatrix}.
\]

As there are no dipoles in the drift, the \( 1/\rho \) term in Eq. (19) is zero and we get the extended 3 × 3 matrix

\[
M_{\text{Drift}} = \begin{pmatrix}
    1 & \ell & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 1
\end{pmatrix}.
\]

To calculate the dispersion in a FODO cell we refer back to the thin-lens approximation used for the beta-function calculation. The matrix for a half-cell has been derived in Eq. (10). In thin-lens approximation, the length \( \ell \) of the drift is just half the length of the cell, as the quadrupole lenses have zero length.

\[
M_{\text{half Cell}} = \begin{pmatrix}
    C & S \\
    C' & S'
\end{pmatrix} = \begin{pmatrix}
    1 - \frac{\ell}{\rho} & \ell \\
    \frac{-\ell}{\rho^2} & 1 + \frac{\ell}{\rho}
\end{pmatrix}.
\]

Using this expression we can calculate the terms \( D, D' \) of the 3 × 3 matrix

\[
D(s) = S(s) \star \int_{\sigma_0}^{S(s)} \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(s) \star \int_{\sigma_0}^{S(s)} \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}
\]

\[
D(\ell) = \ell \star \frac{1}{\rho} \star \int_{0}^{\ell} \left(1 - \frac{s}{\rho}\right) ds - \left(1 - \frac{\ell}{\rho}\right) \star \frac{1}{\rho} \int_{0}^{\ell} ds
\]
In an analogous way one derives the expression for $D'$

$$D'(\ell) = \frac{\ell}{\rho} \left( 1 + \frac{\ell}{2f} \right)$$

and we get the complete matrix for a FODO half-cell

$$M_{\text{halfCell}} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{\ell}{f} & \frac{\ell}{\rho} & \frac{\ell^2}{2\rho} \\ -\frac{\ell}{f^2} & \frac{\ell}{f} & \frac{\ell}{\rho} \left( 1 + \frac{\ell}{2f} \right) \\ 0 & 0 & 1 \end{pmatrix}.$$  

![Figure 13: Beta function (upper part). Dispersion (lower part) function in a FODO cell.](image)

Because of symmetry, the dispersion in a FODO lattice reaches its maximum value in the centre of a QF quadrupole and its minimum in a QD, as shown in Fig. 13 where, in addition to the $\beta$ function, the dispersion is shown in the lower part of the figure.

Therefore we get the boundary conditions for the transformation from a QF to a QD lens

$$\begin{pmatrix} \dot{D} \\ 0 \\ 1 \end{pmatrix} = M_{1/2} * \begin{pmatrix} \dot{D} \\ 0 \\ 1 \end{pmatrix}$$
to calculate the dispersion at these locations.

\[ D = \dot{D}(1 - \frac{\ell}{f}) + \frac{\ell^2}{2\rho} \]

\[ 0 = -\frac{\ell}{f^2} \dot{D} + \frac{\ell}{\rho}(1 + \frac{\ell}{2f}) \] .

Remember that we have to use the focal length of a half-quadrupole

\[ \tilde{f} = 2f \]

and that the phase advance is given by

\[ \sin(\varphi/2) = \frac{L_{\text{Cell}}}{2\tilde{f}} \]

For maximum dispersion in the middle of a focusing quadrupole and for the minimum dispersion in the middle of a defocusing lens, we get the expressions

\[ \dot{D} = \frac{\ell^2}{\rho} (1 + \frac{1}{2} \sin \frac{\varphi}{2}) \]

\[ D = \frac{\ell^2}{\rho} (1 - \frac{1}{2} \sin \frac{\varphi}{2}) \] .

Note that dispersion depends on the half-length \( \ell \) of the cell, the bending strength of the dipole magnet \( 1/\rho \), and the phase advance \( \varphi \).

The dependence of \( D \) on the phase advance is shown in the plot of Fig. 14. Both values \( D_{\text{max}} \) and \( D_{\text{min}} \) decrease for an increasing phase \( \varphi \) (which is just another way to say ‘for increasing focusing strength’, as \( \varphi \) depends on the focusing strength of the quadrupole magnets).

To summarize:
– small dispersion needs strong focusing and therefore large phase-advance;
– there is, however, an optimum phase advance concerning the best (i.e. smallest) value of the \( \beta \) function;
– the stability criterion limits the choice of the phase advance per cell.
Dispersion at the focusing and defocusing quadrupole lens in a FODO as a function of the phase $\phi$

It is necessary to find a compromise for the focusing strength in a lattice that takes into account the stability of the motion, the $\beta$ function in both transverse planes, and the dispersion function. In a typical high-energy machine, this optimization is not too difficult as the dispersion does not impact much on the beam parameters (as long as it is compensated at the interaction point of the two beams).

In synchrotron light sources, the beam emittance is usually the parameter to be optimized (that means in nearly all cases minimized), and as the emittance depends on the dispersion $D$ in an electron storage ring, the dispersion function and its optimization is of greatest importance in these machines.

The horizontal beam emittance in an electron ring is given by the expression

$$\varepsilon_x = \frac{55}{32\sqrt{3}} \frac{h}{mc^2} \gamma^2 \frac{1}{J_x} \frac{H(s)}{\left\langle \frac{1}{R^2} \right\rangle}$$

where the function $H(s)$ is defined by

$$H(s) = \gamma D^2 + 2\alpha DD' + \beta D'^2.$$ 

The optimization of $H(s)$ in a magnet structure is a subject of its own and an introduction to that field of so-called low-emittance lattices can be found in the contribution of A. Streun in this school [9].

2.5 Orbit distortions in a periodic lattice
The lattice we have designed consists of a small number of basic elements.

Bending magnets define the geometry of the circular accelerator and for a given particle momentum, the size of the machine. Quadrupole lenses define the phase advance of the single particle trajectory and through this parameter define the beam dimensions and the stability of the motion.

To fill the empty spaces in the lattice cell with some useful other components, we have to talk about the 'O-s' of the FODO.
Field errors in storage rings have been discussed in the linear beam optics lecture and we know that in the case of a dipole magnet an error of the bending field is described by an additional kick $\delta$ (typically measured in mrad) on the particles

$$\delta = \frac{ds}{\rho} = \int \frac{Bds}{p/e} .$$

The beam oscillates in the corresponding plane and the resulting orbit amplitude is

$$x(s) = \sqrt{\beta(s)} \sqrt{1 + \frac{1}{\rho(s)} \cos(|\varphi(s) - \varphi(s)| - \pi Q)} ds .$$

This is given by the beta function at the place of the dipole magnet $\beta(s)$ and its bending strength $1/\rho$, and the beta function at the observation point in the lattice $\beta(s)$.

For the lattice designer that means if a correction magnet has to be installed in the lattice cell, it should be placed at a location where $\beta$ is high in the corresponding plane.

At the same time Eq. (21) tells us that the amplitude of an orbit distortion is highest at the place where $\beta$ is high and this is the place where beam-position monitors have to be located to measure the orbit distortion precisely.

In practice, both beam-position monitors and orbit-correction coils are located at places in the lattice cell where the $\beta$ function in the considered plane is large, i.e. close to the corresponding quadrupole lens.

### 2.6 Chromaticity in a FODO cell

The chromaticity $\zeta$ describes an optical error of a quadrupole lens in an accelerator. For a given magnetic field, i.e. the gradient of the quadrupole magnet, particles with smaller momentum will feel a stronger focusing force and vice versa.

The chromaticity $\zeta$ relates the resulting tune shift to the relative momentum error of the particle

$$\Delta Q = \zeta \frac{\Delta p}{p}$$

and as it is a consequence of the focusing properties of the quadrupole magnets, it is given by the characteristics of the lattice. For small momentum errors $\Delta p/p$ the focusing parameter $k$ can be written

$$k(p) = \frac{g}{p/e} = \frac{e}{p_0 + \Delta p}$$

where $g$ denotes the gradient of the quadrupole lens, $p_0$ the design momentum, and the term $\Delta p$ refers to the momentum error. If $\Delta p$ is small as we have assumed, we can write

$$k(p) \approx \frac{e}{p_0} \left(1 - \frac{\Delta p}{p} \right) = k + \Delta k .$$

This describes a quadrupole error

$$\Delta k = -k_0 \frac{\Delta p}{p}$$
and leads to a tune shift of

\[
\Delta Q = \frac{1}{4\pi} \int \Delta k^* \beta(s) ds
\]

\[
\Delta Q = -\frac{1}{4\pi} \frac{\Delta p}{p} \int k_s \beta(s) ds .
\]

By definition, the chromaticity \(\xi\) of a lattice is given by

\[
\xi = -\frac{1}{4\pi} \int \beta(s) k(s) ds .
\] (22)

Assuming the accelerator consists of \(N\) identical FODO cells then, replacing the \(\beta(s)\) by its maximum value at the focusing and its minimum value at the defocusing quadrupoles permits the approximation of the integral by the sum

\[
\xi \approx -\frac{1}{4\pi} N^* \frac{\hat{\beta} - \beta}{f_Q} ;
\]

\(f_Q = 1/(k^* f)\) denotes the focal length of the quadrupole magnet.

\[
\xi \approx -\frac{1}{4\pi} N^* \frac{1}{f_Q} \left\{ \frac{L(1+\sin \varphi/2) - L(1-\sin \varphi/2)}{\sin \varphi} \right\} ;
\] (23)

uses the expressions from Eq. (15) for \(\beta^*\) and \(\hat{\beta}\). With some useful trigonometric transformations like

\[
\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}
\]

we can transform the right-hand side of (23) to get

\[
\xi = -\frac{1}{4\pi} N^* \frac{1}{f_Q} \frac{L^* \sin \varphi/2}{\sin \varphi/2 \cos \varphi/2}
\]

or for one single cell, \(N = 1\),

\[
\xi_{Cell} = -\frac{1}{4\pi} \frac{1}{f_Q} \frac{L^* \tan \varphi/2}{\sin \varphi/2} .
\]

Remembering the relation

\[
\sin \frac{\varphi}{2} = \frac{L}{4f_Q}
\]
we obtain a surprisingly simple result for the chromaticity contribution of a FODO cell.

\[ \xi_{\text{cell}} = -\frac{1}{\pi} \tan \frac{\phi}{2}. \]

3 LATTICE INSERTIONS

In Fig. 6, the lattice of a typical machine for the acceleration of high-energy particles consists of two quite different parts: The arcs built out of a number of identical cells, and the straight sections that connect them and that house complicated systems like dispersion suppressors, mini-beta insertions, or high-energy particle detectors.

3.1 Drift space

For an insight into the design of lattice insertions, it is useful to look at a simple drift space embedded in a normal lattice structure.

What happens to the beam parameters \( \alpha, \beta \) and \( \gamma \) if we stop focusing for a while? The transfer matrix for the Twiss parameters from a point '0' to position 's' in a lattice is given by the formula

\[
\begin{pmatrix}
\beta \\
\alpha \\
\gamma
\end{pmatrix}_s =
\begin{pmatrix}
C^2 & -2SC & S^2 \\
-CC' & SC' + S'C & -SS' \\
C'^2 & -2'S'C' & S'^2
\end{pmatrix}
\begin{pmatrix}
\beta \\
\alpha \\
\gamma
\end{pmatrix}_0
\]

where the cosine and sine functions \( C \) and \( S \) are given by the focusing properties of the lattice elements between the two points.

For a drift space of length \( s \) this is according to Eq. (4) as simple as

\[
M = \begin{pmatrix}
C & S \\
C' & S'
\end{pmatrix} = \begin{pmatrix}
1 & s \\
0 & 1
\end{pmatrix}
\]

and the optical parameters develop as a function of \( s \)

\[
\beta(s) = \beta_0 - 2\alpha s + \gamma_0 s^2 \\
\alpha(s) = \alpha_0 - \gamma_0 s \\
\gamma(s) = \gamma_0.
\]

We now examine the relations more closely.

3.1.1 Location of a beam waist

The first equation indication that if the drift space is long enough, even a convergent beam at position '0' will become divergent as the term \( \gamma_0 s^2 \) is always positive, see Fig. 15.
Therefore there will be a point in the drift where the beam dimension is smallest, in other words where the beam envelope has a waist. The position of this waist can be calculated by

$$\alpha(s_i) = 0$$

the second equation of (25) then gives

$$\alpha_0 = \gamma_0 * s_1$$

or

$$s_1 = \frac{\alpha_0}{\gamma_0}$$.

The position of the waist is given by the ratio of the $\alpha$- and $\gamma$-function at the beginning of the drift.

As the $\gamma$ parameter is constant in a drift and $\alpha$ is zero at the waist, we can directly calculate the beam size that we get at the waist.

$$\gamma(s_i) = \gamma_0, \quad \alpha(s_i) = 0$$

$$\beta(s_i) = \frac{1 + \alpha^2(s_i)}{\gamma(s_i)} = \frac{1}{\gamma_0}$$.

The beta function at the location of the waist is given by the inverse of the gamma function at the beginning of the drift; a nice, simple scaling law.

### 3.1.2 Beta function in a drift space

To expand on the behaviour of the Twiss parameters in a drift, the scaling of $\beta$ with the length of the drift has a large impact on the design of high-energy machines. Assuming that we are in the centre of a drift and the situation is symmetric, index ‘0’ refers to the position at the starting point, but we now want to have a left–right symmetric optics with respect to it, $\alpha_0 = 0$. From Eq. (25) at the starting point we get

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

and as we know from Eq. (26) that at the waist $s = 0$, we have

$$\gamma_0 = \frac{1 + \alpha_0^2}{\beta_0} = \frac{1}{\beta_0}$$

and we get $\beta$ as a function of the distance $s$ from the starting point:
\[ \beta(s) = \beta_0 + \frac{s^2}{\beta_0}. \]  

(27)

I would like to point out two facts in this context:

– The relation (27) is a direct consequence of Liouville’s theorem: The density of the particle’s phase space is constant in an accelerator. In other words, with conservative forces only, beam emittance \( \varepsilon \) is constant, leading immediately to the relation (24). The conservation of \( \varepsilon \) is a fundamental law that cannot be avoided and there is no way to overcome the increase of the beam dimension in a drift.

– The behaviour of \( \beta \) in a drift has a strong impact on storage-ring design. As large beam dimensions have to be avoided, large drift spaces are forbidden or at least very inconvenient. In Section 3.2, we see that this is one of the major limitations of the luminosity of colliding beams in an accelerator.

At the beam waist we can derive another short relation often used for scaling beam parameters. The beam envelope \( \sigma \) is given by the beta function and the emittance of the beam,

\[ \sigma(s) = \sqrt{\varepsilon \beta}, \]

and the divergence \( \sigma' \) by

\[ \sigma'(s) = \sqrt{\varepsilon \gamma}. \]

Now, as \( \gamma = (1 + \alpha^2)/\beta \), wherever \( \alpha = 0 \) the beam envelope has a local minimum (i.e. a waist) or maximum. At that position the beta function is just the ratio of the beam envelope and the beam divergence.

\[ \beta(s) = \frac{\sigma(s)}{\sigma'(s)} \quad \text{at a waist}. \]

If we cannot fight against Liouville’s theorem, we can at least try to optimize its consequences. Equation (27) for \( \beta \) in a symmetric drift can be used to find the starting value that gives the smallest beam dimension at the end of the drift of length \( \ell \).

Setting

\[ \frac{d\hat{\beta}}{d\beta_0} = 1 - \frac{\ell^2}{\beta_0^2} = 0 \]

gives us the value of \( \beta_0 \) that leads to the smallest \( \beta \) after a drift of length \( \ell \):

\[ \beta_0 = \ell. \]

(28)

For a starting value of \( \beta_0 = \ell \), the maximum beam dimension at the end of the drift will be smallest and its value is just double the length of the drift

\[ \hat{\beta} = 2\beta_0 = 2\ell. \]

### 3.2 Mini-beta insertions

Section 3.1 showed that the \( \beta \) function in a drift space can be chosen with respect to the length \( \ell \) to minimize the beam dimension and accordingly the aperture requirements for vacuum chambers and magnets. In general, the value of \( \beta \) is in the order of a few metres and the typical length of drift spaces in a lattice is of the same order.
However, the straight sections of a storage ring are often designed for the collision of two counter-rotating beams. The beta functions at the collision points are therefore very small compared to their values in the arc cells. Typical values are more in the range of centimetres than of metres. Still, the same scaling law of Eq. (28) holds and the optimum length of such a drift would, for example, be \( \ell \approx 36 \text{ cm} \) for the interaction regions of the two beams in the HERA collider.

Modern high-energy detectors, in contrast, are impressive devices consisting of many large components that do not fit into a drift space of a few centimetres. Figure 16 shows the ZEUS detector at the HERA collider. Obviously, to install a huge detector like this, the storage-ring lattice needs special treatment.

![Particle detector of the ZEUS collaboration at the HERA storage ring](image)

**Fig. 16:** Particle detector of the ZEUS collaboration at the HERA storage ring

The lattice has to be modified before and after the interaction point, to establish a large drift space where the high-energy experiment detector can be embedded. At the same time, the beams have to be strongly focused to get very small beam dimensions in both transverse planes at the collision point, in other words to get high luminosity. This lattice structure is called ‘mini-beta insertion’.

The luminosity of a particle collider is defined by the event rate \( R \) of a special reaction (e.g. a particle produced by beam collision).

\[
R = \sigma_R \cdot L .
\]

The production rate \( R \) of a reaction is given by its cross-section \( \sigma_R \) and a number that is the result of the lattice design: the luminosity \( L \) of the storage ring. It is given by the beam optics at the collision point and the amount of the stored beam currents [10].

\[
L = \frac{1}{4\pi e^2 f_0 b} \cdot \frac{I_1 \ast I_2}{\sigma_x \ast \sigma_y} .
\]

Here \( I_1 \) and \( I_2 \) are the values of the stored beam currents, \( f_0 \) is the revolution frequency of the machine, and \( b \) the number of stored bunches. \( \sigma_x \) and \( \sigma_y \) in the denominator are the beam sizes in the horizontal and vertical plane at the interaction point. For a high-luminosity collider, the stored beam currents have to be high and at the same time the beams have to be focused at the interaction point to very small values.
Figure 17 shows the typical layout of a mini-beta-insertion scheme. It generally consists of
– a symmetric drift space large enough to house the particle detector and whose beam waist $\alpha_0 = 0$ is centred at the interaction point of the colliding beams;
– a quadrupole doublet (or triplet) on each side as close as possible;
– additional quadrupole lenses to match the Twiss parameters of the mini-beta insertion to
the optical parameters of the lattice cell in the arc.

As a mini-beta scheme is always a kind of symmetric drift space, we can apply the formula
derived above. For $\alpha = 0$, we get a quadratic increase of the beta function in the drift and at the
distance $\ell_1$ of the first quadrupole lens we get

$$\beta(s) = \beta_0 + \frac{\ell^2}{\beta_0}.$$  

The size of the beam at the position of the second quadrupole can be calculated in a similar way.
According to Fig. 18, the transfer matrix of the quadrupole doublet system consists of four parts:
Two drifts with lengths $\ell_1$ and $\ell_2$ and a focusing and defocusing quadrupole magnet. Starting at
the IP we get — again in thin-lens approximation:

$$M_{D_1} = \begin{pmatrix} 1 & \ell_1 \\ 0 & 1 \end{pmatrix}, \quad M_{f_1} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f_1} & 1 \end{pmatrix},$$

$$M_{D_2} = \begin{pmatrix} 1 & \ell_2 \\ 0 & 1 \end{pmatrix}, \quad M_{f_2} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix}.$$
In general the first lens of this system focuses in the vertical plane and therefore according to the sign convention used in this School the focal length is positive, \( 1/f_i > 0 \). The matrix for the complete system is

\[
M = M_{QF}^* M_{D2}^* M_{D1}^* M_{QD}^* M_{D1}
\]

Multiplying out we get

\[
M = \begin{pmatrix}
1 & 0 & 1 / f_i & 1 / f_2 & 1 / f_1
0 & 1 & 1 / f_1 & 1 & 1 / f_2
-1 / f_2 & 1 & 0 & 1 / f_1 & 1
1 & 0 & 1 & 1 / f_1 & 1
\end{pmatrix}
\]

Remembering the transformation of the Twiss parameters in terms of matrix elements [see Eq. (6)]

\[
\begin{pmatrix}
\alpha
\beta
\gamma
\end{pmatrix}_s = \begin{pmatrix}
C^2 & -2CS & S^2 \\
-C'C & SC' + CS' & -SS'
C'^2 & -2S'C' & S'^2
\end{pmatrix}_0 \begin{pmatrix}
\alpha
\beta
\gamma
\end{pmatrix}
\]

we put in the above terms and obtain

\[
\beta(s) = C^2 \beta_0 - 2SC \alpha_0 + S^2 \gamma_0 .
\]

Here the index ‘0’ denotes the interaction point and ‘s’ refers to the position of the second quadrupole lens. As we are starting at the IP where \( \alpha_0 = 0 \) and \( \gamma_0 = 1 / \beta_0 \), we can simplify and get

\[
\beta(s) = \beta_0 + S^2 / \beta_0
\]

This formula for \( \beta \) at the second quadrupole lens is very useful when the gradient and aperture of the mini-beta quadrupole magnet have to be designed.
3.2.1 Phase advance in a mini-beta insertion

Unlike the situation in the arc where the phase advance is a function of the focusing properties of the cell, in a mini-beta insertion or in any long drift space it is, for the most part, a constant: As we know from linear beam optics, the phase advance is given by

$$\Phi(s) = \int \frac{1}{\beta(s)} ds$$

and inserting $\beta(s)$ from Eq.(27), we get

$$\Phi(s) = \frac{1}{\beta_0} \int_0^{\ell_1} \frac{1}{1 + s^2 / \beta_0^2} ds$$

$$\Phi(s) = \arctan \frac{\ell_1}{\beta_0}$$

where $\ell_1$ denotes the distance of the first focusing element from the IP, i.e. the length of the first drift space. In Fig. 19 we have plotted the phase advance as a function of $\ell$ for a $\beta$ function of 10 cm. If the length of the drift is large compared to the value of $\beta$ at the IP, which is usually the case, the phase advance approximates 90 degrees on each side. In other words, the tune of the accelerator will increase by half an integer within the complete drift space.

![Graph showing phase advance vs. drift length](image)

**Fig. 19:** Phase advance in a symmetric drift space as a function of the drift length

There are more remarks concerning mini-beta sections that we shall mention briefly. As we have seen, large values of the beta function on either side of the interaction point cannot be avoided if a mini-beta section is inserted in the machine lattice. These high-$\beta$ values have a strong impact on the machine performance:

- According to Eq. (22), the chromaticity of a lattice is given by the strength of the focusing elements and the value of the $\beta$ function at that position. In a mini-beta insertion, unfortunately, we have both strong quadrupoles and large beam dimensions.

$$\xi = \frac{-1}{4\pi} \int \{k(s)\beta(s)\} ds$$

The contribution of this lattice section to $\xi$ can be very large, and as it has to be corrected in the ring, it places strong limitations on the luminosity in a collider ring.

- During insertion the beam dimension can reach high values, the aperture of the doublet magnets has to be much larger than in the FODO structure of the arc. Large magnet...
apertures limit the strength of the quadrupole. A compromise has to be found between aperture requirement, integrated focusing strength, spot size at the IP, and money, as large magnets are quite expensive.

- Last but not least is the problem of field quality and adjustment. Compared to the standard magnets in the arc, the lenses in a mini-beta section have to fulfil stronger requirements. A kick, caused by a dipole error or due to an off-centre quadrupole lens, leads to an orbit distortion proportional to the beta function at the place of the error [Eq. (21)].

The field quality concerning higher multipole components has to be much higher and the adjustment of the mini-beta quadrupoles much more precise than those of quadrupole lenses in the arc. In general, multipole components of the order \( \Delta B/B = 10^{-4} \) with respect to the main field and alignment tolerances in the transverse plane of about a tenth of a millimetre are desired.

### 3.2.2 Guidelines for the design of a mini-beta insertion

1. The periodic solution of the lattice cell in the arc has to be calculated to provide starting values for the insertion.

2. Introduce the drift space needed for the insertion device (e.g. the particle detector).

3. Put the mini-beta quadrupoles as close as possible to the IP—nowadays these lenses are often embedded in the detector to keep the distance \( s \) small.

4. Introduce additional quadrupole lenses to match the optical parameters of the insertion to the solution of the arc cell. In general, functions \( \alpha_x, \beta_x, \alpha_y, \beta_y \) and horizontal dispersion \( D_x, D'_x \) have to be matched. Sometimes, additional quadrupoles are needed to adjust the tunes in both planes, in the case of HERA even the vertical dispersion \( D_y, D'_y \) needs to be corrected. So, at least eight additional magnet lenses are necessary.

### 4 DISPERSION SUPPRESSORS

The dispersion function \( D(s) \) has already been mentioned in Section 2.4 where we have shown that it is a function of the focusing properties of the lattice cell, and where we have calculated its size as a function of the cell parameters.

Returning to this topic in the context of lattice insertions: In the interaction region of an accelerator, the straight section of a ring where two counter-rotating beams collide (usually designed as a mini-beta insertion) the dispersion function \( D(s) \) has to disappear. A non-vanishing dispersion dilutes the luminosity of the machine and leads to additional stop bands in the working diagram of the accelerator (synchro-betatron resonances), driven by the beam–beam interaction.

Therefore sections have to be inserted in our magnet lattice to reduce the function \( D(s) \) to zero, these are known as dispersion suppressing schemes. In Eq. (18) we have shown that the oscillation amplitude of a particle is given by

\[
x(s) = x_{p0}(s) + D(s) \frac{\Delta p}{p_0},
\]

where \( x_{p0} \) describes the solution of the homogeneous differential equation (valid for particles with ideal momentum \( p_0 \)) and the second term — the dispersion term — describes the additional oscillation amplitude for particles with a relative momentum error \( \Delta p/p_0 \).
For example, take some numbers from the HERA proton storage ring: The beam size at the collision point of the two beams is in the horizontal and vertical direction given by the mini-beta insertion: $\sigma_x \approx 118\,\mu\text{m}$ and $\sigma_y \approx 32\,\mu\text{m}$. The contribution of the dispersion function to the particles’ oscillation amplitude with a typical dispersion in the cell of $D(s) \approx 1.5\,\text{m}$ and a momentum distribution of the beam $\Delta p/p \approx 5\times10^{-4}$ amounts to $x_{13} = 0.75\,\text{mm}$.

4.1. Dispersion suppression using additional quadrupole magnets: ‘the straightforward way’

There are several ways to suppress the dispersion, each with advantages and disadvantages, which are not covered in this paper. However, the rationale of the basic idea is presented. Let us assume a periodic lattice is given and one simply wants to continue this FODO structure of the arc through the straight section — but with vanishing dispersion.

Given an optical solution in the arc cells as shown in Fig. 20, we have to guarantee that starting from the periodic solution of the optical parameters $\alpha(s)$, $\beta(s)$, and $D(s)$ we obtain a situation at the end of the suppressor where we get $D(s) = D'(s) = 0$ and the values for $\alpha$ and $\beta$ unchanged.

The boundary conditions

\[ D(s) = D'(s) = 0 \]
\[ \beta_x(s) = \beta_x^{\text{arc}}, \quad \alpha_x(s) = \alpha_x^{\text{arc}} \]
\[ \beta_y(s) = \beta_y^{\text{arc}}, \quad \alpha_y(s) = \alpha_y^{\text{arc}} \]

can be fulfilled by introducing six additional quadrupole lenses whose strengths have to be matched individually in an adequate way. This can be done using one of the beam-optics codes available in every accelerator laboratory. An example is shown in Fig. 21, starting from a FODO structure with a phase advance of $\phi \approx 61^\circ$ per cell.

The advantages of this scheme are

- it works for arbitrary phase advance of the arc structure;
- matching works also for different optical parameters $\alpha$ and $\beta$ before and after the dispersion suppressor;
- the ring geometry is unchanged as no additional dipoles are needed.

Fig. 20: Periodic FODO including the horizontal dispersion function in the lower part of the plot
On the other hand there are a number of disadvantages:

- As the strength of the additional quadrupole magnets have to be matched individually, the scheme needs additional power supplies and quadrupole magnet types which can be an expensive requirement.
- The required quadrupole fields are in general stronger than those in the arc.
- The $\beta$ function reaches higher values (sometimes really high values) and so the aperture of the vacuum chamber and of the magnets has to be increased.

There are alternative ways to suppress the dispersion which do not need individually powered quadrupole lenses but instead change the strength of the dipole magnets at the end of the arc structure.

4.2 ‘The clever way’: half-bend schemes

This dispersion-suppressing scheme exits of $n$ additional FODO cells that are added to the periodic arc structure but where the bending strength of the dipole magnets is reduced. As before, we split the lattice into three parts: the periodic structure of the FODO cells in the arc, the lattice insertion where the dispersion is suppressed, followed by a dispersion-free section which can be another FODO structure without bending magnets or a mini-beta insertion etc.

The calculation of the suppressor requires several steps.

4.2.1 Establish the matrix for a periodic arc cell

We have already calculated the dispersion in a FODO lattice in Eq. (20). In thin-lens approximation we have derived a formula for $D$ as a function of the focusing properties of the lattice. Now we have to be a little bit more accurate and instead of the focusing strength and phase advance we have to work with the optical parameters of the system. We know that the transfer matrix in a lattice of a storage ring can be written as a function of the optical parameters of Eq. (14).
\[
M_{0 \to S} = \begin{pmatrix}
\sqrt{\frac{\beta_s}{\beta_0}} (\cos \phi + \alpha_0 \sin \phi) & \sqrt{\beta_s \beta_0} \sin \phi \\
(\alpha_0 - \alpha_s) \cos \phi - (1 + \alpha_0 \alpha_s) \sin \phi & \sqrt{\beta_s \beta_0} (\cos \phi - \alpha_s \sin \phi)
\end{pmatrix}.
\]

(29)

The variable \( \Phi \) refers to the phase advance between the starting point ‘0’ and the end point ‘s’ of the transformation. The formula is valid for any starting and end point in the lattice. If, for convenience, we refer the transformation to the middle of a focusing quadrupole magnet (as we usually did in the past) where \( \alpha = 0 \), and if we are interested in the solution for a complete cell, we can write the equation in a simpler form. Extending the matrix to the 3 \times 3 form to include the dispersion terms [see Section (2.4)] and taking into account the periodicity of the system, \( \beta_0 = \beta_s \), we get

\[
M_{\text{Cell}} = \begin{pmatrix}
C & S & D \\
C' & S' & D'
\end{pmatrix} = \begin{pmatrix}
\cos \phi_c & \beta_c \sin \phi_c & D(l) \\
-\frac{1}{\beta_c} \sin \phi_c & \cos \phi_c & D'(l)
\end{pmatrix}.
\]

(30)

Now \( \phi_c \) is the phase advance for a single cell and the index ‘c’ reminds us that we talk about the periodic solution (one complete cell).

The dispersion elements \( D \) and \( D' \) are as usual given by the \( C \) and \( S \) elements according to Eq. (19)

\[
D(\ell) = S(\ell)^* \int_0^\ell \frac{1}{\rho(\bar{s})} C(\bar{s}) d\bar{s} - C(\ell)^* \int_0^\ell \frac{1}{\rho(\bar{s})} S(\bar{s}) d\bar{s}
\]

\[
D'(\ell) = S'(\ell)^* \int_0^\ell \frac{1}{\rho(\bar{s})} C(\bar{s}) d\bar{s} - C'(\ell)^* \int_0^\ell \frac{1}{\rho(\bar{s})} S(\bar{s}) d\bar{s}.
\]

The values \( C(\ell) \) and \( S(\ell) \) refer to the symmetry point of the cell (the middle of the quadrupole). The integral, however, has to be taken over the dipole magnet, where \( \rho \neq 0 \). Assuming a constant bending radius in the dipole magnets of the arc, \( \rho = \text{const} \) (which is a good approximation in general), we can solve the integral over \( C(s) \) and \( S(s) \) if we approximate their values by those in the middle of the dipole magnet.

**Fig. 22:** Schematic view of a FODO: notation of the phase relations in the cell
4.2.2 Transformation of the optical functions from the centre of the quadrupole to the middle of the dipole, to calculate the \( C(\hat{s}) \) and \( S(\hat{s}) \) functions

As indicated in Fig. 22 we have to transform the optical functions \( \alpha \) and \( \beta \) from the centre of the quadrupole lens to the centre of the dipole magnet. The formalism is given by Eq (29) and we get (with \( \alpha_0 = 0 \))

\[
C_m = \sqrt{\frac{\beta_m}{\beta_C}} \cos \Delta \phi = \sqrt{\frac{\beta_m}{\beta_C}} \cos \left( \frac{\phi_c}{2} \pm \varphi_m \right)
\]

\[
S_m = \sqrt{\beta_m \beta_C} \sin \left( \frac{\phi_c}{2} \pm \varphi_m \right).
\]

The index \( m \) tells us that we are now dealing with values in the middle of the bending magnets and as our starting point was the centre of the QF quadrupole, the phase advance for this transformation is half the phase advance of the cell, which brings us to the QD lens, plus/minus the phase distance \( \varphi_m \) from that point to the dipole centre.

Now we can solve the integrals for \( D(s) \) and \( D'(s) \)

\[
D(\ell) = S(\ell) \int_0^{\ell} \frac{1}{\rho(\hat{s})} C(\hat{s}) d\hat{s} - C(\ell) \int_0^{\ell} \frac{1}{\rho(\hat{s})} S(\hat{s}) d\hat{s}
\]

\[
D(\ell) = \beta_C \sin \phi_c \frac{L_B}{\rho} \sqrt{\frac{\beta_m}{\beta_C}} \cos \left( \frac{\phi_c}{2} \pm \varphi_m \right) - \beta_C \frac{L_B}{\rho} \sqrt{\beta_m \beta_C} \sin \left( \frac{\phi_c}{2} \pm \varphi_m \right).
\]  \hspace{1cm} (31)

With \( L_B \) the length of the dipole magnets and putting \( \delta = L_B/\rho \) for the bending angle we get

\[
D(\ell) = \delta \sqrt{\beta_m \beta_C} \left\{ \sin \phi_c \left[ \cos \left( \frac{\phi_c}{2} + \varphi_m \right) + \cos \left( \frac{\phi_c}{2} - \varphi_m \right) \right] - \cos \phi_c \left[ \sin \left( \frac{\phi_c}{2} + \varphi_m \right) + \sin \left( \frac{\phi_c}{2} - \varphi_m \right) \right] \right\}.
\]

Using the trigonometric relations

\[
\cos x + \cos y = 2 \cos \frac{x + y}{2} \cos \frac{x - y}{2},
\]

\[
\sin x + \sin y = 2 \sin \frac{x + y}{2} \cos \frac{x - y}{2},
\]

we get

\[
D(\ell) = \delta \sqrt{\beta_m \beta_C} \left\{ \sin \phi_c \left[ 2 \cos \frac{\phi_c}{2} \cos \varphi_m - \cos \phi_c \left( \sin \frac{\phi_c}{2} \cos \varphi_m \right) \right] \right\}
\]

\[
D(\ell) = 2 \delta \sqrt{\beta_m \beta_C} \cos \varphi_m \left\{ \sin \phi_c \cos \frac{\phi_c}{2} - \cos \phi_c \sin \frac{\phi_c}{2} \right\}
\]

and with
\[
\sin 2x = 2 \sin x \cdot \cos x \\
\cos 2x = \cos^2 x - \sin^2 x
\]

we can derive the dispersion at the middle of the quadrupole magnet in its final form

\[
D(\ell) = 2\delta \sqrt{\beta_m / \beta_C} \cdot \cos \varphi_m \left\{ 2 \sin \frac{\phi_C}{2} \cdot \cos^2 \frac{\phi_C}{2} - \left( \cos^2 \frac{\phi_C}{2} - \sin^2 \frac{\phi_C}{2} \right) * \sin \frac{\phi_C}{2} \right\}
\]

\[
D(\ell) = 2\delta \sqrt{\beta_m / \beta_C} \cdot \cos \varphi_m \cdot \sin \frac{\phi_C}{2} \left\{ 2 \cos^2 \frac{\phi_C}{2} - \cos^2 \frac{\phi_C}{2} + \sin^2 \frac{\phi_C}{2} \right\}
\]

\[
D(\ell) = 2\delta \sqrt{\beta_m / \beta_C} \cdot \cos \varphi_m \cdot \sin \frac{\phi_C}{2}.
\]  \hspace{1cm} (32)

This is the expression for the dispersion term of matrix (30) at the centre of the quadrupole magnet, determined from the dipole strength \(1/\rho\) and matrix elements \(C\) and \(S\) at the position of the dipole.

In full analogy, one derives the formula for the derivative of the dispersion, \(D'(s)\)

\[
D'(\ell) = 2\delta \sqrt{\beta_m / \beta_C} \cdot \cos \varphi_m \cdot \cos \frac{\phi_C}{2}.
\]  \hspace{1cm} (33)

As this refers to the situation in the middle of a quadrupole, the expressions for \(D(s)\) and \(D'(s)\) are valid for a periodic structure, namely one FODO cell. Therefore, we require periodic boundary conditions for the transformation from one cell to the next:

\[
\begin{pmatrix}
D_C \\
D'_C
\end{pmatrix} = M_C \cdot \begin{pmatrix}
D_C \\
D'_C
\end{pmatrix}
\]

and by symmetry

\[
D'_C = 0.
\]  \hspace{1cm} (34)

With these boundary conditions the periodic dispersion in the FODO cell is determined.

\[
D_C = D_C \cdot \cos \phi_C + \delta \sqrt{\beta_m / \beta_C} \cdot \cos \varphi_m \cdot 2 \sin \frac{\phi_C}{2}
\]

\[
D_C = \delta \sqrt{\beta_m / \beta_C} \cdot \cos \varphi_m / \sin \frac{\phi_C}{2}.
\]  \hspace{1cm} (35)

### 4.2.3 Calculate the dispersion in the suppressor part

In the dispersion suppressor section, \(D(s)\) starting with the value at the end of the cell is reduced to zero. Or turning it around and thinking from right to left: The dispersion has to be created, starting from \(D = D' = 0\). The goal is to generate the dispersion in this section so that the values of the periodic arc cell are obtained.
The relation for $D(s)$ still holds

$$D(\ell) = S(\ell) \star \int_0^1 \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(\ell) \star \int_0^1 \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

but now we can take several cells into account, (the number of cells inside the suppressor scheme) and we will have the freedom to choose a dipole strength $\rho_{\text{supr}}$ in this section, which differs from the strength of the arc dipoles. As the dispersion is generated in a number of $n$ cells the matrix for these $n$ cells is

$$M_n = M_C^n = \begin{pmatrix} \cos n\phi_C & \beta_C \sin n\phi_C & D_n \\ -1/\beta_C \sin n\phi_C & \cos n\phi_C & D'_n \\ 0 & 0 & 1 \end{pmatrix}$$

and according to (31) the dispersion created in these $n$ cells is given by

$$D_n = \beta_C \sin n\phi_C \delta_{\text{supr}} \sum_{i=1}^n \cos \left( i\phi_C - \frac{1}{2} \phi_m \pm \phi_m \right) \sqrt{\beta_m/\beta_C} -$$

$$- \cos n\phi_C \delta_{\text{supr}} \sum_{i=1}^n \sqrt{\beta_m/\beta_C} \sin \left( i\phi_C - \frac{1}{2} \phi_m \pm \phi_m \right)$$

$$D_n = \sqrt{\beta_m/\beta_C} \sin n\phi_C \delta_{\text{supr}} \sum_{i=1}^n \cos \left( \frac{(2i-1)}{2} \phi_m \pm \phi_m \right) -$$

$$- \sqrt{\beta_m/\beta_C} \delta_{\text{supr}} \cos n\phi_C \sum_{i=1}^n \sin \left( \frac{(2i-1)}{2} \phi_m \pm \phi_m \right).$$

Remembering the trigonometric gymnastics shown above we get

$$D_n = \delta_{\text{supr}} \sqrt{\beta_m/\beta_C} \sin n\phi_C \sum_{i=1}^n \cos \left( \frac{(2i-1)}{2} \phi_m \right) * 2 \cos \phi_m -$$

$$- \delta_{\text{supr}} \sqrt{\beta_m/\beta_C} \cos n\phi_C \sum_{i=1}^n \sin \left( \frac{(2i-1)}{2} \phi_m \right) * 2 \cos \phi_m$$

$$D_n = 2\delta_{\text{supr}} \sqrt{\beta_m/\beta_C} \cos \phi_m \left\{ \sum_{i=1}^n \cos \left( \frac{(2i-1)}{2} \phi_m \right) * \sin(n\phi_m) - \right\}$$

$$- \sum_{i=1}^n \sin \left( \frac{(2i-1)}{2} \phi_m \right) * \cos(n\phi_m) \right\}$$
\[ D_n = 2\delta_{\text{supr}} \sqrt{\beta_n \beta_C} \cos \varphi_m \sin(n\phi_C) \sin \left( \frac{n\phi_C}{2} \right) \]

\[ -2\delta_{\text{supr}} \beta_n \beta_C \cos \varphi_m \cos(n\Phi_C) \sin \left( \frac{n\Phi_C}{2} \right) \]

\[ D_n = \frac{2\delta_{\text{supr}} \sqrt{\beta_n \beta_C} \cos \varphi_m}{\sin \frac{\phi_C}{2}} \left\{ 2 \sin \frac{n\phi_C}{2} \cos \frac{n\phi_C}{2} \cos \frac{n\phi_C}{2} \sin \frac{n\phi_C}{2} - \ight. \\

\left. - \left( \cos^2 \frac{n\phi_C}{2} - \sin^2 \frac{n\phi_C}{2} \right) \sin^2 \frac{n\phi_C}{2} \right\} . \]

And finally

\[ D_n = \frac{2\delta_{\text{supr}} \sqrt{\beta_n \beta_C} \cos \varphi_m \sin^2 n\phi_C}{\sin \frac{\phi_C}{2}} . \tag{36} \]

This relation gives us the dispersion \( D(s) \) that is created in a number of \( n \) cells with a phase advance of \( \Phi_C \) per cell. \( \delta_{\text{supr}} \) is the bending strength of the dipole magnets located in these \( n \) cells and the optical function \( \beta_m \) and \( \beta_C \) refer to the values at the centre of the dipole and the quadrupole, respectively.

In a similar calculation we get the expression for the derivative \( D'(s) \) of the dispersion:

\[ D'_n = \frac{2\delta_{\text{supr}} \sqrt{\beta_n \beta_C} \cos \varphi_m}{\sin \frac{\phi_C}{2}} \sin n\phi_C . \tag{37} \]

4.2.4 Determine the strength of the suppressor dipoles

The last step is to calculate the strength of the dipole magnets in the suppressor section. As for the optimum match of \( D \) the dispersion generated in this section has to be equal to that of the arc cells we equate the expressions (34),(35) and (36),(37) and get for \( D_n \) the condition

\[ D_n = \frac{2\delta_{\text{supr}} \sqrt{\beta_n \beta_C} \cos \varphi_m \sin \phi_C}{\sin \frac{\phi_C}{2}} \sin^2 \frac{n\phi_C}{2} = \delta_{\text{arc}} \sqrt{\beta_n \beta_C} \cos \frac{\varphi_m}{\sin \frac{\phi_C}{2}} \]

and for \( D' \)
\[
D'_n = \frac{2\delta_{\text{supr}} \sqrt{\frac{\beta_m}{\beta_c}} \cos \varphi_m}{\sin \phi_c} \sin n\phi_c = 0.
\]

From these last equations we deduce two conditions for the dispersion matching
\[
\begin{align*}
2\delta_{\text{supr}} \sin \left(\frac{n\phi_c}{2}\right) &= \delta_{\text{arc}} \\
\sin(n\phi_c) &= 0.
\end{align*}
\]

If the phase advance per cell in the arc fulfils the condition \(\sin(n\Phi_C)=0\), the strength of the dipoles in the suppressor region is just half the strength of the arc dipoles. In other words the phase has to fulfil the condition
\[
\phi_c = k\pi, \quad k = 1, 3, \ldots.
\]

There are a number of possible phase advances that fulfil that relation, but clearly not every arbitrary phase is allowed. Possible constellations would be \(\phi_C = 90^\circ, n = 2\) cells or \(\phi_C = 60^\circ, n = 3\) cells in the suppressor.

Figure 23 shows such a half-bend dispersion suppressor, starting from a FODO structure with 60° phase advance per cell. The focusing strength of the FODO cells before and after the suppressor are identical, with the exception that — clearly — the FODO cells on the right are ‘empty’, i.e. they have no bending magnets.

Obviously, the beta function is now unchanged in the suppressor region, unlike the suppressor scheme with quadrupole lenses.

Again this scheme has advantages:
- no additional quadrupole lenses are needed and no individual power supplies;
- the aperture requirements are just the same as in the arc, as the \(\beta\) functions are unchanged;

and disadvantages:
- it works only for certain values of the phase advance in the structure and therefore restricts the free choice of the optics in the arc;
– special dipole magnets are needed (with half the strength of the arc types);
– the geometry of the ring is changed.

I want to mention, for purists, that in these equations the phase advance of the suppressor part is equal to the one of the arc structure — which is not completely true as the weak focusing term $1/\rho^2$ in the arc FODO differs from the term $1/(2\rho)^2$ in the half-bend scheme. As, however, the impact of the weak focusing on the beam optics can be neglected in many practical cases, Eq. (38) is nearly correct.

The application of this scheme is very elegant, but it has to be embedded in the accelerator design at an early stage as it has a strong impact on the beam optics and geometry.

4.3 The missing-bend dispersion suppressor scheme

Another suppressor scheme is used in a number of storage rings: It consists of a number of $n$ cells without dipole magnets at the end of the arc, followed by $m$ cells identical to the arc cells. The matching condition for this ‘missing-bend scheme’ with respect to the phase advance is

$$\frac{2n+m}{2} \phi_c = (2k+1) \frac{\pi}{2}$$

and for the number $m$ of the required cells.

$$\sin \frac{m \phi_c}{2} = \frac{1}{2}, \quad k = 0, 2 \quad \text{or} \quad \sin \frac{m \phi_c}{2} = -\frac{1}{2}, \quad k = 1, 3 \ldots$$

An example that is based on $\Phi = 60^\circ$ and $m = n = 1$ is shown in Fig. 24.

![Fig. 24: Dispersion suppressor based on the missing-magnet scheme](image)

There are more scenarios for a variety of phase relations in the arc and the corresponding bending strength needed to reduce $D(s)$, see Refs. [11,12].

In general, combine one of the two schemes (missing or half-bend suppressor) with a certain number of individual quadrupole lenses to guarantee the flexibility of the system with respect to phases changes in the lattice and to keep the size of the $\beta$ function moderate.
References