Lattices for light sources

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Abstract
This paper is intended to provide a guideline on how to design the lattice (i.e., the magnet arrangement) of a high brightness light source.

In the introductory section we will consider the general framework to the design, clarify the steps of the design process and the interfaces to technical engineering, the lattice designer has to take into account.

In the second section we will examine the lattice building blocks, i.e., the magnets. A short detour to magnet design will help to understand the limitations of magnet strength.

An introduction to beam dynamics is not the subject of this paper, instead all relevant formula to understand subsequent sections are summarized in section three without any derivation.

Section four focusses on emittance and how to build a lattice for obtaining low emittance. This is first explained in a most intuitive way, then the theoretical minimum emittance and deviations from the minimum conditions are calculated in some details.

The final section dealing with the problem of acceptance considers the requirements for beam life time and injection efficiency. Physical acceptances will be discussed in some details, whereas the large subject of dynamic acceptance optimization is subject of another lecture [25] and only briefly summarized here.

1 Introduction
1.1 Global requirements
Synchrotron light users demands specify the layout of a new light source:

– The highest photon energy to be obtained from the most advanced insertion devices available (in terms of period length and harmonic content) sets the electron energy of the storage ring.
– The target performance in terms of brightness defines the natural, horizontal emittance of the electron beam, and – considering diffraction limited experiments in the X-ray range – also the emittance coupling, i.e., the ratio of vertical to horizontal emittance.
– The number and types of planned beamlines defines the shape of the machine, i.e., the number and length of straight sections and the symmetry of the lattice.
– Experiments expect micron photon beam stability at the location of the probe to be analyzed. Thermal stability of the beamline is maintained by constant photon heat load, i.e., by constant beam current. This requires a long beam life time and/or operation with top-up injection (i.e., very frequent refill).
– Low radiation background to the experiments also calls for long beam lifetime and for efficient injection into the machine, in particular in top-up operation. Both constraints require large acceptances of the machine in order to keep particles deviating in longitudinal and transverse momenta after scattering with residual gas atoms or each other, and in order to safely capture injected beams.
– The available area for building the machine and the beamlines limits the circumference of the lattice. This makes it more challenging to accommodate the requested number of straights while achieving the desired performance, and defines the type of magnetic arc to be used.
– Every light source needs some operational flexibility and some upgrade potential for new and yet unforeseeable future experiments.
– Finally, the limitations of budgets for construction and operation call for most simple and efficient solutions (in terms of number of magnets and magnet families, power consumption, etc.).

1.2 Lattice design phases and tools

The lattice design process may be divided into four phases:

1. **Preparation**: Definition of performance issues and boundary conditions. Acquisition of information on available building blocks (magnets) for composing the lattice concerning their properties and technical limitations.

2. **Linear lattice layout**: Arrangement of linear building blocks (quadrupoles and bending magnets) to obtain the desired global (i.e., concerning the lattice as a whole) quantities like circumference, emittance, etc. This phase deals with the concepts of periodic cells, matching sections, insertions etc.

3. **Nonlinearities**: Introduction of sextupoles and RF cavities for stabilization of particles with momentum deviations. Due to the nonlinearity of these elements dynamic acceptance, i.e., stability limits for transverse and longitudinal deviations from the reference orbit becomes the main design issue.

4. **Errors**: Investigation of lattice performance in presence of magnet misalignments, multipolar errors, vibrations etc. and development of correction schemes. This final design phase ends with a significant prediction on the performance of the machine, resp. tolerance requirements for the components.

Little of lattice design can be done analytically, most tasks require the aid from a computer code. During the linear layout, it is most important for the designer to have a visual, dialog-oriented code in order to ‘play’ with lattices and optimize them interactively, whereas in the later design stages the exact modelling including nonlinearities and errors is crucial in order to obtain predictive results. Typically, work proceeds by alternating fast, creative steps using reduced models, and slow, consolidating runs based on more complete models. Usually not all tasks can be fulfilled by one codes and switching between two or more is required. – Here we list some keywords for lattice development code requirements:

– **Model**: complete set of elements, correct methods for tracking and concatenation, well documented approximations.

– **Elementary functions**: beta functions and dispersions, periodic solutions, closed orbit finder, energy variations, tracking, matching.

– **Toolbox**: Fourier transforms of particle data (see resonance analysis), minimization routines (see dynamic aperture optimization, coupling suppression), linear algebra package (see orbit correction).

– **User convenience**: editor functions, graphical user interface, editable text files.

– **Extended functionality**: RF dimensioning, geometry plots, lifetime calculations, injection design, alignment errors, multipolar errors, ground vibrations.

– **Connectivity**: database access, control system access (see real machine operation).
1.3 Interfacing

Lattice design provides an empty machine, telling how single particles move around the ring and what equilibrium will form for an ensemble of particles. However, prediction of performance in terms of brightness for light sources also includes the maximum beam current to be stored, which is a complex subject to be shared between beam dynamics, vacuum and RF departments. In the framework of this paper it will be assumed as given and not considered further. However, the lattice designer has to stay in contact with his/her colleagues from other departments for several reasons:

- **Vacuum**: Impedance of vacuum chamber affects maximum beam current, pressure affects lifetime, pumps, absorbers and flanges require space.
- **Radiofrequency**: RF parameters determine momentum acceptance and other parameters affecting directly the lattice design.
- **Diagnostics**: Beam position monitors have to be inserted at the appropriate locations (betatron phases) and also require space.
- **Magnet Design**: Technological limits and geometrical properties of magnets determine the maximum magnet strength to be used in lattice design, magnet coils need space, multipolar errors affect the acceptance.
- **Alignment**: Misalignments cause orbit distortions affecting the performance and requiring correction schemes.
- **Mechanical engineering**: Grouping of magnets on stiff girders improves the robustness of the lattice to misalignments and vibrations.
- **Construction**: All devices to be installed by the different departments meet on the design engineer’s blueprint and conflicts will reveal there.

Most important for the lattice designer is to include the space requirements for all devices right from the beginning. Figure 1 illustrates this by comparing the lattice designer’s and the design engineer’s view of a lattice section.

1.4 Conventions and approximations

A curvilinear coordinate system is commonly used for describing particle motion in a storage ring, where the axis $x$ points radially to the ring outside, the $s$-axis is the tangent to the ring, pointing forward, and the $y$-axis points up. Some authors define it as a right handed system $\{x; y; s\}$ [2], other authors as a right
handed system \{x; s; y\} \[4, 20\] (which also might be viewed as a left handed \{x; y; s\} system!). The second convention has the advantages, that particles in the storage ring rotate mathematically positive (i.e., counter clockwise seen from above), and that for constant radius of curvature \(\rho\) the system maps to a cylindrical coordinate system \{\(r; \theta; z\)\} with \(r = \rho + x, \quad \theta = s/\rho, \quad z = y\) \[23\].

Positive particles require a negative magnetic field \(\vec{B} = -|B| \hat{y}\) to get the appropriate Lorentz force to form the storage ring. Since the radius of curvature of a curve in space is always positive by definition, the product of magnetic field and radius of curvature, also called magnetic rigidity \((B\rho)\) (see Eq. (2) below) has to be negative.

For the practical design of a low emittance light source usually an idealized subset of the theoretical beam dynamics formalism is sufficient and obtained by introducing the following approximations:

- Highly relativistic beam, i.e., \(v = c, E = pc\)
- Small deviations from the reference axis \((x, y \ll \rho, x', y' \ll 1)\) for validity of linear beam dynamics using betatron amplitudes, emittance, betafunctions, betatron phases, etc.
- Decoupling of subspaces: Synchrotron motion, i.e., dynamics in \((\delta, \Delta s)\) subspace is slow and thus treated as a constant parameters over the timescales of betatron motion (‘adiabatic approximation’). Coupling between horizontal \((x, x')\) and vertical \((y, y')\) subspaces is considered to be small.
- Nonlinearities are treated as perturbations.

2 Building blocks

Elementary building blocks for linear lattice design are bending magnets and quadrupoles for guiding and focusing the beam. The nonlinear lattice design also includes sextupoles used for correction of the quadrupoles’ chromatic aberrations. The real lattice with alignment and other errors also includes small corrector dipoles and skew quadrupoles as well as beam position monitors. In addition every ring needs injection devices (kickers and septa) and one or more RF cavities for acceleration and longitudinal focusing. Light sources also contain wigglers and undulators for production of highest flux, resp. brightness synchrotron radiation.

2.1 Lattice composition: local \(\leftrightarrow\) global

It is very important to make a clear distinction between local properties of the building blocks and global properties of the lattice as pronounced by Forest and Hirata \[10\]:

‘A quantity is called local if it is derivable from the individual magnet irrespective of the magnet position in the ring and even irrespective of the ring itself. For example a trajectory of a particle through the magnet is local. […] Global information, on the contrary, is derivable only after the full ring is produced. For example the dynamic aperture has no meaning whatsoever if we cannot iterate the one-turn map (i.e., circulate particles in the machine).’

Concatenation of building blocks is done by coordinate transformations, i.e., translations and rotations. For example a vertical bending magnet can be described by a horizontal bending preceded by a 90° rotation around the longitudinal axis.

A building block may have any coordinate system, however in practise it is either a cartesian geometry with parallel entrance and exit planes (\(\{x, y\}\) planes, perpendicular to \(s\)) and length \(L\) or a cylindric geometry with an angle \(\phi\) between the entrance and axis planes and are length \(L\). Obviously cylindric geometry is more convenient for the description of bending magnets and cartesian geometry for other magnet types and drift spaces.
After assembling all building blocks and closing the ring the one turn map can be calculated: It is the mapping of the particle vector \( \vec{X} = (x, x', y, y', \delta, \Delta s) \) from one turn to the next \( \vec{X}_{n+1} \) with the order of the map corresponding to the highest power in the coordinates. Thus the closed orbit is a fixed point of the one-turn map and the transfermatrix is a linearization of the one-turn map around the closed orbit. The so called ‘design orbit’ is just a coincidence of the closed orbit with (most) magnets’ symmetry axis for the ideal lattice but is not defined a priori.

Eventually any lattice design has to be tested by tracking particles through the lattice, since tracking concatenates all local transformation of the particle vector from block entrance to block exit including the coordinate transformations between blocks, and thus implicitly applies the full one-turn map.

2.2 Magnet multipole definition

In the local coordinate system of a magnet the field is given as multipole expansion around the local reference axis \((x=y=0)\) by

\[
B_y(x, s, y) + i B_x(x, s, y) = (B\rho) \sum_n (i a_n(s) + b_n(s))(x + iy)^{n-1}
\]

with \(n\) the multipole order and \(2n\) the number of poles in the magnet, i.e., \(n = 1, 2, 3, \ldots\) indicating dipole, quadrupole, sextupole, etc. The \(b_n\) are the regular multipoles \((B_x = 0\) for \(x = 0\)) and \(a_n\) the skew multipoles, obtained through a rotation around the \(s\)-axis by \(90s/n\).

The quantity \((B\rho)\) is called the magnetic rigidity. From the Lorentz force equation it is directly derived as ratio of momentum over charge:

\[
(B\rho) = -\frac{p}{q} = -\frac{\beta E/e}{n_e c} \approx 3.3536 E [\text{GeV}] \text{ for relativistic electrons } (n_e = -1)
\]

with \(n_e\) the number of elementary charges per particle and \(\beta = v/c\). By differentiation of Eq. (1) we obtain a useful expression for a pure, regular multipole:

\[
b_n = \frac{1}{B\rho (n-1)!} \frac{\partial^{(n-1)} B_y(x, y)}{\partial x^{n-1}} \bigg|_{y=0}.
\]

The radius of a cylinder around the symmetry axis touching the magnet poles is called the pole inscribed radius or magnet aperture radius \(R\). In case of dipoles the full gap \(g = 2R\) is used for characterization. The poletip field of a regular magnet is then given from Eqs. (1) and (3) by

\[
B_{pt} = (B\rho) b_n R^{n-1} = \frac{R^{n-1}}{(n-1)!} \frac{\partial^{(n-1)} B_y(x, y)}{\partial x^{n-1}} \bigg|_{y=0}.
\]

Following the conventions explained in Section 1.4 above, a positive dipole moment \(b_1\) bends both positive and negative charged particles to the ring inside, because the polarity is contained in \((B\rho)\), or other speaking, \(b_1\) is always positive otherwise we have no ring with a coordinate system \(\{x; s; y\}\) oriented as described above. Consequently, a positive quadrupole moment \(b_2\) focuses all particles horizontally.

Unfortunately there are different definitions of multipole strengths: The quadrupole strength, mostly called \(k\), usually is defined as \(k = -b_2\), however \(k = +b_2\) may also be found. For the sextupole (and higher multipole) strength, called \(m\) or \(k_s\) or other, definitions with and without the factorial are used: \(m = \mp b_3\) or \(m = \mp 2b_3\).

2.3 The general bending magnet

The bending magnet is a block of cylindrical symmetry with a reference radius of curvature \(\rho_{\text{ref}}\), an arc length \(L\) and a bending angle \(\Phi = L/\rho_{\text{ref}}\). The dipole moment \(b_1 = B_y/(B\rho)\) provides a radius \(\rho = 1/b_1\)
of a particle’s trajectory’s curvature. The magnetic field is adjusted that \( \rho = \rho_{\text{ref}} \) for the particular energy of the reference particle. For particles at other energies the trajectories’ curvatures do not match the coordinate system’s curvature, they thus leave the bend off-axis even if they entered on-axis, an effect called dispersion. It is important not to mix \( \rho_{\text{ref}} \) which is given by geometry and \( \rho \) which is a function of magnet current and particle energy.

The general magnet may also contain a gradient (quadrupole moment) \( b_2 \) and is called combined function magnet, because it provides both bending and focussing. The transfermatrix for propagating a vector of local coordinates from the entry plane to the exit plane is given by

\[
\begin{pmatrix}
  x \\
  x' \\
  y \\
  y' \\
  \delta
\end{pmatrix}_{\text{out}} =
\begin{pmatrix}
  c_x & \frac{1}{\sqrt{K}} s_x & 0 & 0 & \frac{b_2}{K} (1 - c_x) \\
  -\sqrt{K} s_x & c_x & 0 & 0 & 0 \\
  0 & 0 & c_y & \frac{1}{\sqrt{-b_2}} s_y & 0 \\
  0 & 0 & -\sqrt{-b_2} s_y & c_y & 0 \\
  0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  x \\
  x' \\
  y \\
  y' \\
  \delta
\end{pmatrix}_{\text{in}}
\]

with the abbreviations

\[
c_x[s_x] = \cos[\sin](\sqrt{K} L), \quad c_y[s_y] = \cos[\sin](\sqrt{-b_2} L) \text{ and } K = b_2^2 + b_2, \quad \delta = \Delta p/p_0.
\]

For \( 0 > b_2 > -b_2^2 \) the gradient bend provides horizontal and vertical focussing, for lower values of \( b_2 \) it becomes horizontally defocussing, for positive \( b_2 \) vertically defocussing. (Note: \( \cos ix = \cosh x, \quad \sin ix = i \sinh x \) ) Focussing means negative matrix elements \( m_{21} \) and \( m_{43} \), i.e., a positive value of \( x \), resp. \( y \) provides a negative increment to \( x' \), resp. \( y' \).

For \( b_2 = 0 \), i.e., no gradient, it is a pure sector dipole magnet. For \( b_1 = 0 \), it is a quadrupole (for \( b_1 \rightarrow 0, \rho \rightarrow \infty \) the cylindrical symmetry becomes cartesian) and will not produce any dispersion. If \( b_1 = 0 \) and \( b_2 = 0 \) it is just a drift space.

Weak focusing synchrotrons used combined function magnets, the definition of the ‘field index’

\[
n = -\frac{\rho}{B_y} \frac{\partial B_y}{\partial x} = -\frac{b_2}{b_1^2}
\]

goes back to these times. Later, combined function magnets with strong gradients \((|b_2| \gg b_1^2)\) became again attractive for low emittance lattices [16].

In a pure sector bend the entrance and exit edges of the magnet are orthogonal to the arc, in the general case the edges may be rotated by angles \( \zeta_1, \zeta_2 \). Rectangular bends have parallel entrance and exit edges, \( \zeta_1 = \zeta_2 = \phi/2 \). Laminated magnets like used in synchrotrons for reasons of eddy current suppression are always rectangular since manufacturing is done by stacking the laminates.

### 2.4 Magnet design

A brief detour into magnet design is required in order to include limitations on magnet strengths and requirements for distances between magnets into the lattice design process:

**Length** The effective length \( L_{\text{eff}} \) of the field of an iron dominated magnet of yoke length \( L_{\text{iron}} \) is approximately given by

\[
L_{\text{eff}} := \int \frac{B(s) \, ds}{B_o} \approx L_{\text{iron}} + \frac{2R}{n}
\]

where \( B_o \) is the maximum field in the magnet’s center, \( R \) the pole inscribed radius and \( n \) the multipole order. \( L_{\text{eff}} \) and \( B_o \) are the relevant quantities for beam optics. For the total length \( L_{\text{total}} \) the coil size has to be added to the iron length. As an example consider the dipole magnet shown in Fig. 2: The required coil cross section area is given by \( A = Bg/(2j_c \mu_o) \). A conservative value for the gross average
current density in a water cooled coil (including water channels, insulations, etc.) is \( j_c \approx 2 \ldots 3 \text{ A/mm}^2 \). Assuming a realistic bending magnet of 1.5 T field and 40 mm gap the cross section is \( A \approx 100 \text{ cm}^2 \) and the coil might be quadratic with 10 cm width and height. The iron length will be shorter than the effective length by \( L_{\text{iron}} \approx (L_{\text{eff}} - \text{gap}) \), but the coil in our example would add 16 cm to the effective magnet length as used in lattice design. Same considerations can be done for quadrupoles and sextupoles. Figure 1 shows the spaces required by the coils. Some data for the effective coil width, i.e., \( (L_{\text{total}} - L_{\text{eff}})/2 \), thus to be added in advance on both sides of a magnet as free space are listed in Table 1.

**Maximum poletip field** Some data for the maximum poletip field for normal conducting iron dominated magnets are also given in Table 1. The poletip field is limited due to saturation effects somewhere in the iron. Although magnet iron saturates fully around 2.2 T it becomes nonlinear at lower fields already. In order to maintain the high field homogeneity required for modern machines and to ensure predictability and reproducibility the magnet design should try to avoid saturation. The limits on poletip fields for quadrupoles and sextupoles are lower then for bending magnets since flux lines are compressed due to pole geometry and higher field values appear somewhere else in the yoke. Saturation of a \( 2n \)-pole will create a parasitic \( 6n \)-pole.

**Magnet apertures** Calculations as done in Fig. 2 can be done for any iron dominated multipole and lead to the result that the required current per coil expressed as (windings \( \times \) current) \( NI \) or (current density \( \times \) coil cross section) \( j_c A \) is proportional to (poletip field \( \times \) aperture radius) \( B_{pt} R \). Keeping the multipole strength constant we thus obtain from Eq. (4) a proportionality of \( NI \propto R^n \), e.g., quadrupole currents increase with the square of aperture. The proportionality is the same for the power if we assume constant current density \( j_c \). Thus from the magnet design point of view the apertures should be as small as possible.

On the other hand the apertures have to be as large as necessary to allow efficient injection and sufficient beam lifetime, and also sufficient cross sections for pumping. Since poletip field is the limiting quantity, larger aperture decreases the maximum acceptable multipole strength. With larger aperture a

![Fig. 2: Iron dominated dipole magnet. Integrating Maxwell’s equation \( \oint H \cdot ds = \int j \cdot da \) along the path as shown at the left gives the cross section area \( A \) of the coils required to create the field \( B: A = B/(2j_c)[S_{\text{iron}}/(\mu_0 \mu_r) + g/\mu_0] \) with \( j_c \) the current density in the coil. Since the iron permeability \( \mu_r \gg 1 \) this simplifies to \( A \approx Bg/(2j_c \mu_0) \). The figure at right shows the distinction between effective magnet length, iron length and total length due to addition of coil width to iron length.

<table>
<thead>
<tr>
<th></th>
<th>eff. coil width</th>
<th>max. poletip field</th>
<th>aperture radius</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bending magnets:</td>
<td>6.5 . . . 15 cm</td>
<td>1.5 T</td>
<td>20 . . . 35 mm (=gap/2)</td>
</tr>
<tr>
<td>Quadrupoles:</td>
<td>4 . . . 7 cm</td>
<td>0.75 T</td>
<td>30 . . . 43 mm</td>
</tr>
<tr>
<td>Sextupoles:</td>
<td>4 . . . 8 cm</td>
<td>0.6 T</td>
<td>30 . . . 50 mm</td>
</tr>
</tbody>
</table>

Table 1: Some data for effective coil width maximum poletip fields and aperture inscribed radii obtained from a survey on existing light source magnets.
quadrupole has to be built longer for maintaining the integrated focusing strength, when the maximum pole tip field has been reached. In this way the whole machine size increases with magnet apertures. Some data for pole inscribed radii are also listed in Table 1.

In particular, a light source in its final configuration will run with several undulators of rather narrow vertical gap (few mm). Consequently the storage ring could be built using bending magnets of correspondingly small aperture in order to reduce operating costs. On the other hand, narrow vacuum chamber give rise to the so called resistive wall instability. Thus the best choice of vertical aperture has to carefully considered in the design of a light source.

3 Lattice properties

For an introduction to linear transverse dynamics we refer to refs. [20, 22, 23, 29] covering the linear Hamiltonian of betatron motion with appropriate approximations (small curvature, paraxial motion, piecewise constant fields), the equations of motion, Hill’s equation and introduction of betafuncions, betatron phases and dispersion, and how to obtain these quantities from a transfer matrix. These references also give examples of transfermatrices for particular magnets and of basic lattice cells like a triplet, a FODO cell, etc., obtained by multiplication of the concatenated magnets’ transfermatrices.

For our practical approach to lattice design only the essential formulae needed for the understanding of the following sections are summarized here:

3.1 Betafunction and emittance

For a linear lattice with negligible transverse coupling and ‘slow’ longitudinal dynamics, the betafuction \( \beta(s) \) is introduced for the following motivations:

- Seen from a practical point of view, the r.m.s. beam size (assuming Gaussian particle distributions as they result from the quantum structure of synchrotron radiation), can be expressed as

\[
\sigma_x = \sqrt{\frac{\text{Betafunction}}{\text{magnet structure}}} \times \sqrt{\frac{\text{Emittance}}{\text{particle ensemble}}}.
\]

The betafunction is completely defined by the magnetic fields no matter if there are particles in the machine or not, only the reference energy has to be specified. The emittance however describes the invariant phase space volume of the particles no matter which focussing forces act on them. (Of course, in long term (thousands of turns) the emittance is also determined by the magnet structure, see Section 4)

- Seen from a theoretical point of view, the single particle Hamiltonian of the betatron oscillation is canonically transformed from coordinates \( x, p_x \) (with \( p_x \approx x'p_0 \)) to action-angle variables, namely the invariant amplitude \( 2J \) and the betatron phase \( \phi(s) \):

\[
H = \frac{p_x^2}{2} + \frac{b_2(s)x^2}{2} \quad \rightarrow \quad \tilde{H} = \frac{J}{\beta(s)}, \quad \epsilon = \langle J \rangle.
\]

Here the beta-function describes the variation of the Hamiltonian along the lattice, and the emittance turns out to be the average particle amplitude [20, 29], (see Section 4.1).

3.2 The betatron oscillation

The linear betatron motion of a particle is described by

\[
x(s) = \sqrt{2Jx \cdot \beta_x(s)} \cos \phi_x(s) + D(s) \cdot \delta
\]
with \( D = D_x \) the dispersion function describing the orbit translation for relative momentum deviations \( \delta = \Delta p/p_0 \). The same equation applies to the vertical \( y \), with \( D_y(s) \equiv 0 \) in case of a flat lattice (i.e., no vertical bending magnets).

\( \alpha, \beta, \gamma \) are called the Twiss parameters — not to be confused with the relativistic parameters! — and related with each other and the betatron phase by

\[
\phi(s) = \int \frac{1}{\beta(s)} \, ds \quad \alpha(s) = -\frac{1}{2} \frac{d\beta(s)}{ds} \quad \gamma(s) = \frac{1 + \alpha(s)^2}{\beta(s)}. \tag{8}
\]

The angle \( x'(s) \) is obtained by differentiation of Eq. (7) using the relations of Eq. (8):

\[
x'(s) = \sqrt{\frac{2J}{\beta_x(s)}} \left( \sin \phi_x(s) + \alpha_x(s) \cos \phi_x(s) \right) + D'(s) \cdot \delta. \tag{9}
\]

### 3.3 Circle transformation

The Poincaré plot \( \{x, x'\} \) at some location \( s \) paints an ellipse in phase space during subsequent turns [perhaps with a dispersive offset of the origin]. This ellipse can be converted into a circle of radius \( \sqrt{2J} \) by the geometric transformation (also see Fig. 3)

\[
\begin{pmatrix} \chi \\ \chi' \end{pmatrix} = T \begin{pmatrix} x \\ x' \end{pmatrix} \quad \text{with} \quad T = \begin{pmatrix} 1/\sqrt{\beta_x} & 0 \\ -1/\sqrt{\beta_x} & \sqrt{\beta_x} \end{pmatrix} \tag{10}
\]

\[
\Rightarrow \begin{pmatrix} \chi \\ \chi' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \cos \phi_x & \sin \phi_x \\ \sin \phi_x & \cos \phi_x \end{pmatrix} \begin{pmatrix} 1/\sqrt{\beta_x} & D/\beta_x \delta \\ -1/\sqrt{\beta_x} & \sqrt{\beta_x} \delta \end{pmatrix}. \tag{11}
\]

### 3.4 Transfer matrix

The general transfer matrix \( M_{a \rightarrow b} \) from some location \( a \) to another location \( b \) in the lattice is conveniently described by a circle transformation at location \( a \), followed by a rotation in phase space by the betatron phase advance \( \Delta \phi = \phi_b - \phi_a \) and by a backtransformation at location \( b \):

\[
M_{a \rightarrow b} = T_b^{-1} \begin{pmatrix} \cos \Delta \phi & \sin \Delta \phi \\ -\sin \Delta \phi & \cos \Delta \phi \end{pmatrix} T_a. \tag{12}
\]

Multiplication gives

\[
M_{a \rightarrow b} = \begin{pmatrix} \frac{\sqrt{A}(\cos \Delta \phi + \alpha_a \sin \Delta \phi)}{\sqrt{\beta_a \beta_b}} \sin \Delta \phi \\ \frac{\sqrt{A}(\cos \Delta \phi - (1+\alpha_a \alpha_b) \sin \Delta \phi)}{\sqrt{\beta_a \beta_b}} \sin \Delta \phi \end{pmatrix}. \tag{13}
\]

For a periodic structure, i.e., \( b = a \), this simplifies to the one-turn matrix

\[
M_a = \begin{pmatrix} \cos 2\pi Q + \alpha_a \sin 2\pi Q & \beta_a \sin 2\pi Q \\ -\gamma_a \sin 2\pi Q & \cos 2\pi Q - \alpha_a \sin 2\pi Q \end{pmatrix}. \tag{14}
\]

with \( Q \) the machine tune, i.e., the number of betatron oscillations per revolution. This matrix further simplifies when considering a symmetry point of the machine where \( \alpha = 0 \).

It is common use to obtain the global transfermatrix numerically by multiplication of the local element matrices, however this method is dangerous, since it works only for the special case, that the elements are lined up in perfect alignment, so that the beam travels along the symmetry axis of every element. This case is usually given in the linear lattice design phase and therefore it works although it is not the correct procedure. In the general case, as mentioned above (see Section 2.1), the transfermatrix is a linearization around the closed orbit which can be anywhere after concatenation of the elements.
3.5 Twiss parameter propagation

The transformation of twiss parameters along the lattice is given by a $3 \times 3$ matrix composed of elements of $M_{a \rightarrow b}$, here expressed in term of sine and cosine type solutions:

$$
\begin{pmatrix}
\beta \\
\alpha \\
\gamma
\end{pmatrix}_b =
\begin{pmatrix}
C^2 & -2SC & S^2 \\
-CC' & S'C + SC' & -SS' \\
C'^2 & -2S'C' & S'^2
\end{pmatrix}
\cdot
\begin{pmatrix}
\beta \\
\alpha \\
\gamma
\end{pmatrix}_a
\text{ with } M_{a \rightarrow b} = \begin{pmatrix}
C & S \\
C' & S'
\end{pmatrix}.
$$

(15)

3.6 Example: Drift

As the most simple example, we may consider a drift space, the $[(x, x')]$ sub-matrix given by $\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ (see Eq. (5) for $b_1 = 0, b_2 = 0$). From a focus, where $\beta = \beta_o, \alpha = 0$ and thus $\gamma = 1/\beta$ (see Eq. (8)), the betafunction propagates (see Eq. (15)) as

$$
\beta(s) = \beta_o + \frac{s^2}{\beta_o}.
$$

(16)

The phase advance of a drift space extending from $-L$ to $+L$ with a focus in the centre, is given by integration of Eq. (8) (left):

$$
\phi = 2 \arctan \frac{L}{\beta_o} \xrightarrow{\beta_o \to 0} 180^\circ = \frac{1}{2} 2\pi.
$$

(17)

The maximum betatron phase advance of a sharp beam focus thus is limited to $180^\circ$, or to 0.5 in tune.

4 Low emittance lattice development

For light sources low emittance is the design criterion, because the brightness of the synchrotron radiation scales quadratically with the inverse emittance in the X-ray region and at least linearly in the [diffraction limited] VUV-region [14].

4.1 Emittance definition and conventions

In an electron storage ring, the two competing synchrotron radiation effects of quantum noise excitation and classical radiation damping lead to a 6-dimensional Gaussian particle distribution with standard deviations $\sigma_x, \sigma_y$ called the [r.m.s.] beam radii, $\sigma_{x'}, \sigma_{y'}$ the beam divergences, $\sigma_\delta$ the relative energy spread.
and $\sigma_s$ the bunch length. Betatron oscillations are rotations in the transverse 2-dimensional \(\{x, x'\}\) and \(\{y, y'\}\) sub phase spaces, which are usually considered as decoupled from each other and adiabatically decoupled from the slow synchrotron oscillation in the longitudinal \(\{\delta, \Delta s\}\) sub phase space.

Figure 3 (right) shows particles in horizontal phase space with a strong and variable correlation between $x$ and $x'$ depending on the local twiss parameters. By means of the circle transformation Eq. (10) normalized coordinates $\chi$, $\chi'$ are introduced as shown in Fig. 3 (left): The corresponding Gaussian distributions have standard deviations which are just given by $\sigma_\chi = \sigma_{\chi'} = \sqrt{\epsilon}$. Obviously the phase space area containing particles up to 1 standard deviation is given by $A = \pi \epsilon$. This remains true for the real coordinates, since the transformation conserves phase space area ($|T| = 1$). Actually, the emittance is invariant to the betatron motion and projects locally varying beam radii and divergences and correlations of both. – In normalized coordinates, the distribution function for the particles is simply

$$\varrho(\chi, \chi') = \frac{1}{2\pi \epsilon} e^{-(\chi^2 + \chi'^2)/(2\epsilon)} .$$

Going back to real coordinates by the inverse transformation $T^{-1}$ (see Eq. (10)), a correlation term appears

$$\varrho(x, x') = \frac{1}{2\pi \epsilon} e^{-(\gamma x^2 + 2\alpha x x' + \beta x'^2)/(2\epsilon)} .$$

The observable, 1-dimensional spatial distribution is obtained by

$$\varrho(x) = \int_{-\infty}^{+\infty} \varrho(x, x') \, dx' = \frac{1}{\sqrt{2\pi \sigma_x}} e^{-x^2/(2\sigma_x^2)} \text{ with } \sigma_x = \sqrt{\epsilon/\beta} .$$

Transforming into action-angle variables $J$ and $\phi$ through Eq. (11) we get

$$\varrho(J, \phi) = \frac{1}{2\pi \epsilon} e^{-J/\epsilon} .$$

It follows, that emittance is the average betatron amplitude of a Gaussian distributed particle ensemble:

$$\langle J \rangle = \int_0^{2\pi} \int_0^{\infty} J \varrho(J, \phi) \, d\phi \, dJ = \epsilon .$$

There exist different conventions for emittance: Electron storage rings quote the 1-sigma-emittance as introduced here. The corresponding area of phase space (grey circle/ellipse in Fig. 3) contains only 39.3% of the particles. An intervall of $[-\sigma, +\sigma]$ in one dimension however confines 68.3% of the particles since it implies integration of the other coordinates over $]-\infty, \infty[$.

Note: For proton machines, a 2-sigma-emittance is quoted, which is four times larger and confines 86.5% of the particles. This fraction is rather independent of the type of distribution, thus making the 2-sigma-emittance a more robust quantity for protons, which are not always Gaussian distributed.

There are also different opinions on whether the phase space area $A$ as shown in Fig. 3 is given by $F = \pi \epsilon$ or by $F = \epsilon$. In the latter case, the factor $\pi$ is included in the emittance unit, measuring it in $\pi \cdot \text{m-rad}$, whereas we here measure emittance in units of $\text{m-rad}$, or, since emittance is very small in light sources, in $\text{nm-rad}$.

Finally, normalized emittance $\tilde{\epsilon}$ refers to true phase space in canonical coordinates \(\{x; p_x\}\). For sub-relativistic beams with large spread of momentum, the difference may be substantial, but for a light source, the simple conversion $\tilde{\epsilon} = m_o c^2 \gamma \epsilon$ is usually justified.

For a deeper understanding of emittance, why it is an invariant, the connection to Liouville’s theorem, its statistical interpretation and the formation of the synchrotron radiation equilibrium see Refs. [3, 7, 21].
4.2 Equilibrium emittance

No matter what has been injected into a storage ring, within a few milliseconds the synchrotron radiation effects will reach an equilibrium and shape the particle ensemble to Gaussian distributions. The natural horizontal emittance being the most relevant of these equilibrium values for a light source, is solely determined by the lattice structure, and for a flat lattice (i.e., only horizontal bending magnets) in practical units given by [21]

$$\epsilon_{xo}[\text{nm.rad}] = 1470 \left( E[\text{GeV}] \right)^2 \frac{\langle H/\rho^3 \rangle}{J_x \langle 1/\rho^2 \rangle}$$

(18)

with $\langle \ldots \rangle$ an average over the lattice, and $H$ the so called ‘lattice invariant’ or ‘dispersion’s emittance’ (actually it is the betatron amplitude of the dispersion see Eq. (11)),

$$H(s) = \gamma_x(s)D(s)^2 + 2\alpha_x(s)D(s)D'(s) + \beta_x(s)D'(s)^2.$$  

(19)

The horizontal damping partition number $J_x$ will be discussed in Section 4.5 below, in most cases it is close to unity, sometimes however intentionally pushed up to values of $\approx 2$ in order to halve the emittance.

In case of an isomagnetic lattice, i.e., all magnets having same bending radius, Eq. (18) simplifies to

$$\epsilon_{xo}[\text{nm.rad}] = 1470 \left( E[\text{GeV}] \right)^2 \frac{\langle H \rangle_{\text{mag}}}{\rho J_x}$$

(20)

with $\langle \ldots \rangle_{\text{mag}}$ an average taken over the magnets only.

At first glance, Eq. (19) tells, that a rather sharp horizontal focus of the beam in each bendig magnet’s center is the path to low emittance: With $\alpha_x = D' = 0$ defining a focus and a low value of dispersion at the focus, $H$ will be small everywhere inside the bending magnets. Before examining $H$ systematically in Section 4.4 to learn how to obtain the minimum emittance, the way to build a low emittance light source lattice will be explained from a more intuitive point of view:

4.3 Building a lattice

To work on periodic structures one may first consider a cell providing a bending angle of $\frac{360^\circ}{N}$ and imagine that it is repeated $N$ times to give a ring. A solution for the betafunction exists, if there is appropriate horizontal and vertical focusing. Fig. 4 displays how to construct a light source lattice cells step by step:

a. Weak focussing: The most simple cell one could think of is one combined function magnet from Eq. (5). However the allowed range of gradients for bounded motion in both transverse planes is restricted to rather small, negative numbers of $b_2$. Therefore the focussing is weak and betafunctions and dispersions will be large resulting in large emittance of the beam. However this type of lattice is interesting for simple and compact industrial light sources which require flux, not brightness, for irradiation of materials [12].

b. Strong focussing: The next simple cell consists of two quadrupoles of opposite polarity: they do not cancel each other but provide focussing in both planes (this is easy to show by multiplication of the matrices). This is the principle of alternating gradient (AG) or strong focussing, such named in contrary to the weak focussing of single combined function magnets: The quadrupole gradients are orders of magnitude larger than the gradients of the weak focussing combined function magnet from the previous example, thus they provide strong variation of betafunctions and sharp foci.

c. The FODO cell: However, a series of quadrupoles gives no ring. So the next simple cell would be alternating quadrupoles with dipole magnets between. The influence of the dipoles on the beam is small compared to the focussing forces from the quadrupoles, wich thus mainly determine the solution for the beta functions. This is the classical FODO cell: [horizontally] focussing...
Fig. 4: The steps of building a low emittance light source lattice. The lenses are quadrupoles, convex if horizontally focussing. The blocks are bending magnets, with an empty lens overplotted if they have a gradient. Dispersion, horizontal beta and vertical beta are indicated by the solid, dashed and dotted lines. See text for further explanation.

quadrupole (F), dipole (no gradient, negligible focussing, 0), defocussing quadrupole (D), dipole (0). FODO cells do not allow low emittance but can be built rather dense, thus they are mainly used for high energy physics machines and booster synchrotrons.

d. **Separated function low emittance cell:** The emittance in the FODO lattice is limited, because the beam has its horizontal focus not in the bending magnets but in the D-quadrupoles. So another type of cell is required which could be called FDODF: With the bending magnet in the symmetry point the horizontal focus is moved to its centre and the emittance can become very low, if the F-quadrupoles are strong enough. Several cells of this type forming a ring would already be a low emittance light source.
e. Combined function low emittance cell: As an alternative to the previous example, the vertical focussing may be provided by the bending magnet, which thus becomes a combined function magnet again. This cell is more compact and requires less elements, also, as will be explained in Section 4.5 below, the horizontal damping partition number \( J_x \) appearing in Eq. (18) may be increased by appropriate choice of the gradient.

f. Low emittance FODO cell: Driving further the previous example, also the strong horizontal focussing may be provided by a combined function magnet. With clever distribution of bending angle, length and gradients on the two types of magnets it is possible to achieve low emittance and appropriate damping partitioning with a very simple lattice [16]. However, the disadvantage is the lack of flexibility since the magnet gradients are usually realized by the pole profiles and thus not adjustable.

g. Dispersion matching: Highest brightness synchrotron light is not obtained from bending magnets but from undulators. These devices usually have negligible focussing, but they are rather long and require suitable empty straight sections for installation, preferably without dispersion in order to hide any energy fluctuation from the users and avoid increase of the source size due to the beam’s energy spread. Thus a transition has to be constructed from the periodically oscillating dispersion to zero dispersion: Going back to the periodic separated function cell of Fig. 4.d, we see, that in the bending magnet’s center the dispersion \( D \) is close to zero and \( D' = 0 \) by symmetry. Thus the periodic cell may be split in the center, leaving only a half bending magnet. Suppressing the dispersion completely to \( D = 0 \) while keeping \( D' = 0 \) can either be done by adjusting the strengths of the F and D quadrupoles, or, more elegantly and smoothly by increasing the distance between the last quadrupole and the half bending magnet, or by a combination of both means.

h. Matching cell: The dispersion suppressing cell is no longer periodic. It is a so called matching cell where some final parameters are to be obtained from some initial parameters. In order to insert this matching cell in a lattice without disturbing the periodic solution, a symmetry point (\( D' = 0 \), \( \alpha_x = 0, \alpha_y = 0 \)) has to be created at its exit. Further, the undulator has to be accomodated in the straight section. The two additional constraints \( \alpha_x = \alpha_y = 0 \) require two more degrees of freedom, i.e., two more quadrupoles. If also a particular value of betafunction (for maximum acceptance, see Section 5.4) or phase advance (for reasons of dynamic aperture optimization, see Section 5.6) is required, three or four quadrupoles are required. Figure 4.h shows a triplet solution for matching the beam to the straight section including the undulator. Appending the mirror image of this complete matching cell, named \( M \) will then form a structure \( [M|\bar{M}] \) which matches the periodic solution from the cell (called \( [P] \)) of Fig. 4.d at the transition points (entry and exit). Thus it can be inserted anywhere in a series of periodic cells: \([\ldots P|P|P|M|\bar{M}|\bar{M}|\bar{M}|\bar{M}|\bar{P}|\bar{P}|\bar{P}|\ldots] \)

With periodic cells \( [P] \) and matching cells \( [M] \) different types of lattices can be composed:

i. Double bend achromat: The structure \( [-M|M] \) is called a double bend achromat (DBA), because it contains two bending magnets, where the first builds up dispersion and the second one suppresses it again, making the whole structure achromatic. Most light sources are of DBA type, for example ESRF, ELETTRA, SOLEIL and others.

j. Triple bend achromat: The structure \( [-M|P|M] \) is called a triple bend achromat (TBA). It is more compact than a DBA lattice of same emittance, however provides fewer straight sections. The TBA can bemodified further by making the half bending magnets at the ends somewhat longer on expense of the center bending magnet in order to obtain the best emittance. Light sources with TBA structure are ALS, PLS and SLS.

\( [-M|P|M] \) is a quadruple bend achromat (QBA) [9], and so on. Any number of \( [P] \) cells could be inserted between \( [-M] \) and \( [M] \), however this option is rather used for damping rings than for light sources, because the number of straight sections becomes too small.
Fig. 5: ESRF and SLS lattices as examples for DBA and TBA structures.
Table 2: Some high brightness light sources in operation. $N_{\text{mag}}$ is the number of bending magnets in the lattice, $\epsilon_{xo}$ [nm rad] is the natural horizontal emittance, and $F$ is the ratio of emittance to its theoretical minimum, see Eq. (21)).

<table>
<thead>
<tr>
<th>Name</th>
<th>Country</th>
<th>$E$ [GeV]</th>
<th>$N_{\text{mag}}$</th>
<th>$\epsilon_{xo}$ [nm rad]</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ALS</td>
<td>USA</td>
<td>1.5</td>
<td>36</td>
<td>3.4</td>
<td>8.9</td>
</tr>
<tr>
<td>MAX-2</td>
<td>Sweden</td>
<td>1.5</td>
<td>20</td>
<td>8.7 (+D)</td>
<td>3.9</td>
</tr>
<tr>
<td>BESSY-2</td>
<td>Germany</td>
<td>1.7</td>
<td>32</td>
<td>5.2</td>
<td>7.5</td>
</tr>
<tr>
<td>ELETTRA</td>
<td>Italy</td>
<td>2.0</td>
<td>24</td>
<td>7.0</td>
<td>3.1</td>
</tr>
<tr>
<td>PLS</td>
<td>S.Korea</td>
<td>2.0</td>
<td>36</td>
<td>12</td>
<td>18</td>
</tr>
<tr>
<td>SLS</td>
<td>Switzerland</td>
<td>2.4</td>
<td>36</td>
<td>5.0</td>
<td>5.1</td>
</tr>
<tr>
<td>ESRF</td>
<td>Europe</td>
<td>6.0</td>
<td>64</td>
<td>4.0 (+D)</td>
<td>3.7</td>
</tr>
<tr>
<td>APS</td>
<td>USA</td>
<td>7.0</td>
<td>80</td>
<td>8.2</td>
<td>11</td>
</tr>
<tr>
<td>Spring-8</td>
<td>Japan</td>
<td>8.0</td>
<td>96</td>
<td>5.6</td>
<td>9.6</td>
</tr>
</tbody>
</table>

(+D) indicates dispersive beams in undulators.

Figure 5 shows the European Synchrotron Radiation Facility ESRF and the Swiss Light Source SLS as examples for DBA and TBA lattices:

The ESRF lattice in its original, dispersion free mode shown in the figure, can be described as $16 \times [-M_H |M_L|-M_L|M_H]$ with two types of matching cells for high and low beta functions in the straight sections. With a 5.625° bending angle for each of the 64 magnets, it provides an emittance of 8.17 nm-rad at 6 GeV beam energy and has a circumference of 846 m.

The structure $3 \times [-M_L |P|M_S|-M_S|P|M_M|-M_M|P|M_S|-M_S|P|M_L]$ describes the lattice of SLS: Three types of matching cells are used for matching to long, medium and short straight sections. As a modification to the basic TBA of Fig. 4.j the ‘half’ bends of the matching cells had been increased on the expense of the center full bend during optimization for lowest emittance. With 8° resp. 14° for the 24 end, resp. 12 center bends, the lattice provides an emittance of 5.03 nm-rad at 2.4 GeV beam energy and has a circumference of 288 m.

### 4.4 Minimum emittance

Solving the integral over $H$ from Eqs. (19) and (20) and minimizing the result in respect to the values of $\alpha_x$, $\beta_x$, $D$ and $D'$ at the magnet centre or entrance gives the theoretical minimum emittance [9,18,26,27].

Assuming that there are identical cells with only one type of bending magnet with deflection angle $\Phi$, and further assuming that $\Phi/2 \ll 1$, which is valid for most light sources ($\Phi < 20^\circ$ gives < 1% error), the emittance can be written as

$$\epsilon_{xo}[\text{nm-rad}] = 1470 \frac{(E[\text{GeV}])^2}{J_x} \frac{\Phi^3 F}{12\sqrt{15}}$$

with $\Phi$ the deflection angle per bending magnet in radian and $F \geq 1$ a factor depending on the lattice type. The theoretical minimum emittance is achieved for $F = 1$. Table 2 gives the $F$ values for some existing high brightness light sources: The region of operation generally is given by $F > 3$.

Note that emittance is independent from the bending radius, resp. the magnetic field, but increases cubically with the angle per bending magnet. That’s why light sources have many cells with relatively short bending magnets.

For two basic situations as shown in Fig. 6 the constraints on beta function and dispersion for obtaining lowest emittance and the minimum factor $F$ can be calculated:

- Center bending magnet: The beam has a focus ($\alpha_{xc} = D'_c = 0$) at the magnet center, dispersion and beta function are symmetric in respect to the bend center and the dispersion is nonzero...
Fig. 6: Requirements to obtain minimum emittance from a center bend (left) or from an end bend (right).

everywhere. Then we get, with \( L = \rho \Phi \) the magnet length:

\[
\beta_{xc} = \frac{1}{2 \sqrt{15}} L \quad D_c = \frac{1}{24 \rho} L^2 \quad \Rightarrow \quad F = 1.
\]

- End bending magnet: The beam enters the bending magnet with zero dispersion. Then we get a constraint for the distance \( s_f \) of the focus (where \( \alpha_x = 0 \)) from the entrance edge and for the betafunction at that focus:

\[
s_f = \frac{3}{8} L \quad \beta_{xf} = \sqrt{\frac{3}{320}} L \quad \Rightarrow \quad F = 3.
\]

It is interesting to note the following:

- Zero dispersion in the center of a bending magnet does not provide the minimum emittance.
- A lattice consisting of only center bends with dispersion everywhere provides the lowest emittance achievable, it is 3 times lower than in a DBA (double bend achromat) lattice where each cell is made from two end bending magnets. As a consequence, DBA lattices starting with dispersion free straights are sometimes later tuned into a dispersive mode in order to further reduce the emittance. However the local effective emittance relevant for the brightness has to include the projection of momentum spread \( \sigma_\delta \) to the horizontal dimension via dispersion and is given by

\[
\epsilon_{x,\text{eff}}(s) = \sqrt{\epsilon_{x0}^2 + \epsilon_{x0} \mathcal{H}(s) \sigma_\delta^2}.
\]  

(22)

- In TBA and higher bend achromats the end magnets should be made shorter by a factor \( \sqrt{3} \) in order to compensate for the factor 3 larger value of \( F \) [15].
- FODO lattices are not suitable for light sources since \( F \approx 100 \), except modern structures with rather different types of combined function magnets used for the F- and D-magnets as shown in Fig. 4.f.
- As to be seen from Table 2 ELETTRA almost operates at the minimum emittance achievable for a DBA lattice with dispersion free straight sections. MAX-2 and ESRF also achieving low \( F \)-values take into account slightly dispersive straights.

Now deviations from the ideal emittance condition will be investigated [13]. This consideration is restricted to a lattice with center bends like shown in Fig. 6 (left). Defining dimensionless parameters

\[
b = \frac{\beta_{xc}}{\beta_{xc,\text{min}}} \quad d = \frac{D_c}{D_{c,\text{min}}}
\]
**Fig. 7:** Ellipses of constant ratios $F$ of emittance to minimum emittance as a function of the deviation from ideal dispersion and $\beta_x$ values for obtaining minimum emittance. Also shown are lines indicating the phase advance per cell [13].

**Fig. 8:** Minimum emittance cell: The 10 o gradient free sector bending magnet with optimum beta functions and dispersion in the center ($b=d=1$) creates the minimum emittance ($E_x$) of 1.5 nm-rad at 3 GeV. The tune advance ($Q_x$) of 0.7902 corresponds to the ideal phase advance of $\Psi = 360^\circ \cdot \Delta Q_x = 284.5^\circ$. 

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with the index \( \min \) denoting the ideal values to obtain the minimum \( F = 1 \), and introducing them into \( H \) after some algebra leads to the equation of an ellipse

\[
\frac{5}{4}(d - 1)^2 + (b - F)^2 = F^2
\]

(23)

shown in Fig. 7. In order to learn more about the cell providing a factor \( F \) we impose constraints on periodicity, i.e., \( \alpha_x = D' = 0 \) at the entrance and exit of cell. Due to symmetry \( \beta_x \) and \( D \) have same values at entrance and exit anyway. In the approximation of small deflection angle \( \Phi/2 \ll 1 \) the matrix \( B \) transforming \( \{x, x'\} \) from center to exit of the bending magnet is given by

\[
B = \begin{pmatrix}
1 & L/2 & \rho \phi^2/8 \\
0 & 1 & \phi/2
\end{pmatrix}
\]

with the third column describing the dispersion production (it is a submatrix from Eq. (5) containing the elements relating \( x, x' \), \( \delta \)). Now the rest of the cell, from bend exit to cell exit, may be described by a matrix \( M \) from which is only known, that it contains no other bending magnet and that it is symplectic of course, i.e., \( |M| = 1 \). Starting with the optical parameters described by \( b \) and \( d \) in the magnet center, the matrix \( M \cdot B \) has to zero \( \alpha_x \) and \( D' \), otherwise the solution would not be periodic. This provides constraints for \( M \). The detailed structure of \( M \) is less interesting than the full cell betatron phase advance \( \Psi \), which is calculated from

\[
\cos \Psi = \frac{1}{2} \text{Trace} \left( M \cdot B \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (M \cdot B)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)
\]

since the first half of the cell is the mirror image of the second half. The result is found as

\[
\Psi = 2 \arctan \left( \frac{6}{\sqrt{15}} \frac{b}{(d - 3)} \right).
\]

(24)

This equation describes lines of constant \( \Psi \) value in the \( (b, d) \) plane intersecting at \( d=3, b=0 \), as shown in Fig. 7. Reaching the minimum emittance requires a phase advance per cell of 284.5°. Ideal lattices based on those cells have been studied [8]: They require ‘empty cells’ alternating with the magnet cells in order to accomodate the additional focus for obtaining high phase advance. An example of a minimum emittance cell is shown in Fig. 8. Existing light sources operate at \( \Psi < 180^\circ \) and accept a larger emittance of \( F \approx 3...5 \), see Table 2.

### 4.5 Damping partitions and energy spread

In Eq. (18) appears the quantity \( J_x \) as another possible way to reduce the emittance. The damping partition numbers are given by

\[
J_x = 1 - D \quad J_y = 1 \quad J_s = 2 + D \quad \text{with} \quad D = \frac{1}{2\pi} \int_{mag} D(s)(b_1(s)^2 + 2b_2(s)) \, ds.
\]

(25)

The integral is only to be taken over the bending magnets where \( b_1 \neq 0 \). If the bending magnets also use large gradients like in some low emittance lattice concepts [16], the gradients have to be adjusted carefully in order to ensure damping, in all dimensions: \(-2 < D < 1\). In a separate function light source lattice with no gradient in the bends, \( D \approx 1 \). Note that \( \sum J_i = 4 \), i.e., the damping may be shifted between the dimensions but the sum is limited. – The damping times depend on the energy loss per turn \( U \) and on the partition numbers:

\[
\tau_i = \frac{2CE}{cU J_i} \quad \text{with} \quad U[keV] = 26.5(E[GeV])^3 B[T]
\]

(26)

for an isomagnetic lattice (neglecting undulator radiation and wake field losses).
The r.m.s. energy (or momentum) spread of the beam is relevant for the effective emittance according to Eq. (22) if undulators are placed in dispersive sections. In practical units it is given by

$$\sigma_\delta = 6.64 \cdot 10^{-4} \cdot \sqrt{\frac{B[T] E[GeV]}{J_s}}.$$  

Thus something should be left for $J_s$ when shifting the partitions around.

### 4.6 Vertical emittance and beam sizes

Ideally, a flat lattice (i.e., no vertical bending magnets) has no vertical emittance, as clear from Eq. (18) with $H_y = 0$ everywhere. In reality however, spurious vertical dispersion and linear coupling create a finite vertical emittance.

Vertical dispersion is caused by roll errors of bending magnets and by vertically misaligned quadrupoles. Linear coupling between horizontal and vertical betatron oscillations is introduced by skew quadrupole fields from vertically misaligned sextupoles or from quadrupoles with roll errors.

If coupling is the main source the emittance ratio $g = \epsilon_y/\epsilon_x$, to be calculated from an integral over skew quadrupolar fields [11], tells how the natural horizontal emittance from eq. Eq. (18) is divided into horizontal and vertical emittance:

$$\epsilon_x = \frac{1}{1 + g \epsilon_{xo}} \epsilon_y = \frac{g}{1 + g} \epsilon_{xo} .$$  

Careful orbit correction including beam based alignment of quadrupoles suppresses vertical dispersion, and dedicated skew quadrupole corrector magnets allow to minimize the linear coupling. Exploiting these two methods emittance ratios $g < 10^{-3}$ can be obtained. However, for VUV and soft X-ray experiments the photon diffraction phase space may be larger than the vertical emittance, thus only hard X-ray experiments profit from lowest $g$-values. On the other side, Touschek lifetime (see Section 5.1) is proportional to the bunch volume and thus scales with $\sqrt{g}$. Therefore, a moderate rather than the minimum achievable value of $g$ is often preferred.

With low coupling and negligible vertical dispersion, the beam will appear as an upright (i.e., not tilted or sheared) elliptical spot with Gaussian profiles and r.m.s. beam radii given by

$$\sigma_x(s) = \sqrt{\epsilon_x \beta_x(s) \left(\sigma_\delta D(s)\right)^2} \quad \sigma_y(s) = \sqrt{\epsilon_y \beta_y(s)} .$$  

For further reading on coupling and beam sizes see refs. [5, 11, 28].

### 4.7 Chromaticity

From Eqs. (2) and (3) we see that the multipole strength $b_n$ is a function of momentum deviation $\delta$,

$$b_n = b_{no} \frac{p_o}{p} = \frac{b_n}{1 + \delta} \approx b_{no} \left(1 - \delta\right) .$$

As a consequence, the quadrupoles set to produce sharp foci in all bending magnets for the sake of low emittance, do not provide the same focusing strength for particles with momentum deviation. Thus the machine tune (total phase advance around the ring) will vary with momentum. The ratio is called chromaticity and given by [7]

$$\xi_x := \frac{dQ_x}{d\delta} = -\frac{1}{4\pi} \int_C b_2(s) \beta_x(s) \, ds \quad \xi_y := \frac{dQ_y}{d\delta} = \frac{1}{4\pi} \int_C b_2(s) \beta_y(s) \, ds .$$  

Chromaticity is naturally negative, i.e., the tunes decrease with energy since the quadrupoles become weaker. Strong quadrupoles at large betafunctions contribute most to chromaticity. Thus light sources suffer from large negative horizontal chromaticity due to the strong focusing. This must not be tolerated for two reasons:
– **Momentum acceptance**: Some variation of momentum has to be accepted by the storage ring for reasons of beam lifetime (see Section 5.1). The machine tune has to stay away from integer or half integer numbers otherwise field or gradient errors will amplify coherently and destroy the beam. Thus even in the optimum case of \( \frac{1}{4\xi} < 1/(4\xi) \) which would give an unacceptable small number for most machines.

– **Head tail instability**: Negative chromaticity excites the fundamental mode of the head tail instability, a collective oscillation of electrons in head and tail of the bunch leading to very fast beam loss. Suppression of the fundamental mode requires non negative chromaticity [6].

Sextupoles in dispersive regions of the lattice are used to compensate the chromaticity: Since a sextupole has a parabolic field \( B_y \sim x^2 \), see Eq. (1)) it may be considered like a quadrupole in the vicinity of a point \( x_d \) by using the tangent: \( b_2(x_d) \approx 2b_3 x_d \). In a dispersive region, the particles are horizontally ‘ordered’ by momentum, i.e., \( x_d = D\delta \). Thus the appropriate combination of dispersion and sextupole strength will compensate the chromatic errors introduced by the quadrupole. Eventually, the chromaticities of a storage ring including sextupoles (but neglecting [small] contributions from bending magnets) are given by [7, 20]

\[
\xi_x = \frac{1}{4\pi} \oint_C (2b_3(s)D(s) - b_2(s))\beta_x(s) \, ds \quad \xi_y = -\frac{1}{4\pi} \oint_C (2b_3(s)D(s) - b_2(s))\beta_y(s) \, ds .
\] (31)

Correcting both chromaticities requires two families of sextupoles with opposite polarities. In order to prevent the two families from counteracting each other, the sextupoles for horizontal chromaticity should be located at locations of large \( \beta_x \) and low \( \beta_y \) and vice versa. Of course, all sextupoles should be located at large dispersion.

### 4.8 Other lattice parameters

While pushing a lattice for lowest emittance, constraints on other lattice parameters have to be taken into account or side-effects have to be minimized:

**Circumference**: For saving space and building costs one would like to make a compact machine. However small circumference is not always the best neither in lattice performance nor in cost. Inserting some space may relax the optics, improve the acceptance, etc. It is also important to take into account from the very beginning of lattice design all kinds of spaces required for magnet coils, beam position monitors, corrector magnets, absorbers, pumps, flanges, etc. as indicated by Fig. 1.

The circumference also must be an integer multiple of the RF wavelength \( C = h\lambda_{rf} \), with \( h \) called the harmonic number. \( h \) could have a nice prime factor decomposition allowing different filling patterns.

**Periodicity**: Periodicity is the number \( N_{\text{per}} \) of identical supercells making up the total lattice (16 for ESRF, 3 for SLS in the example from Fig. 5). High periodicity has fundamental advantages:

– simplicity: most optics calculations are based on the simple supercell;
– stability: few systematic resonances (see below), better dynamic acceptance;
– cost saving: larger series of a few different components.

Periodicity of existing machines ranges from 1 in DORIS to 40 in APS.

**Working point**: The \( \{Q_x, Q_y\} \)-space, called the tune diagram is covered with resonances appearing as lines described by \( aQ_x + bQ_y = p \) as shown in Fig. 9. The resonance order is given by \( a + |b| \). If \( p \) is an integer multiple of \( N_{\text{per}} \), the resonance is systematic, i.e., amplified by the cell structure, else it is inhibited by periodicity. Even \( b \) identifies regular and odd \( b \) skew resonances. The magnets in a flat lattice are all of regular type, thus neither skew nor non-systematic resonances appear in the ideal lattice, but show up in the real lattice including multipolar and misalignment errors (see Fig. 9). – There are many constraints for placing the working point:
Fig. 9: Tune diagram for an ideal (left) and real (right) period-3 lattice. Solid lines are systematic, dashed non-

systematic and dotted skew resonances. Larger thickness corresponds the lower order. Resonances up to $4^{th}$ order 

are shown. The curved line is the tune walk for a wide range of momentum deviations. Here in this example, the 

working point at $20.82/8.28$ turned out to be too close to the $Q_x + Q_y = 29$ sum resonance for the real machine 

and had to be changed.

- It must not be at integer to avoid closed orbit instability due to dipole errors.
- It must not be at half integer to avoid beam blow up due to gradient errors.
- It must not be at a 2$^{nd}$ order sum resonance to avoid mutual amplification of horizontal and vertical 

beatron oscillations.
- It has to stay away from sextupolar resonances (3$^{rd}$ order, see Section 5.6).
- Multiturn injection requires a fractional tune not too close to integer, ($|\text{frac}(Q)| > 0.2$) in the plane 

of injection in order for the injected satellite to clear the septum in the turn following injection (see 

Fig. 10).
- Resistive wall instability requires a tune abover rather than below an integer.

Generally the fractional part of the tune is more important than the integer, but most important is the 

flexibility of the lattice to move the tunes independently in the tune diagram, since the best working 

point eventually is not found till operation. Thus lattice design has to ensure this flexibility.

**Momentum compaction factor** The relative difference in pathlength travelled by a particle at 
given relative momentum deviation $\delta$ within one revolution of the reference particle is expressed by the 
momentum compaction factor $\alpha$:

$$\frac{\Delta s}{C} = \alpha \delta , \quad \alpha = \frac{1}{C} \int_D \frac{D(s)}{\rho} ds$$  \hspace{1cm} (32)

with $C$ the machine circumference. Low emittance light sources have rather low values of $\alpha < 10^{-3}$ 
due to the low dispersion inside the bending magnets. They may become even isochronous, i.e., $\alpha = 0$ 
by introducing partially negative dispersion. At very low or even zero $\alpha$, the quadratic variation of 
pathlength with energy has to be taken into account. It can be controlled by means of sextupoles in order 
to obtain longitudinal stability, i.e., a closed RF bucket again.
5 Acceptance

Closed orbit stability is not enough. The lattice has to accept particles with some deviations from the ideal orbit in all six coordinates $x, x', y, y', \delta$ and $\Delta s$ in order to provide sufficient beam lifetime and to allow filing of the machine to the desired current. Accumulation of beam current requires mainly horizontal acceptance, and increasing the lifetime requires mainly vertical and momentum acceptance.

5.1 Lifetime

Performance in terms of brightness is only useful if the beam lifetime is sufficiently long to keep the beam current approximately constant and to keep the background from lost particles low enough to do the experiments. This requirement is only partially relieved when running top-up injections, because bad lifetime will still lead to enhanced background and to activation of injection elements. The most important processes of particle losses in light sources are:

- Touschek effect is scattering of particles within the bunch, leading to a transfer of transverse to longitudinal momentum exceeding the momentum acceptance. The loss rate is inversely proportional to the bunch volume, thus light sources with their low emittance beams suffer most from Touschek effect.
- Elastic scattering on residual gas nuclei leads to a transverse deflection and subsequent loss in regions of low aperture, which are usually the narrow vertical gaps of the undulators.
- Bremsstrahlung on residual gas nuclei leads to a change of electron momentum and thus also requires sufficient momentum acceptance, however the dependancy is much weaker than for Touschek scattering.

The three relevant lifetime times have the following, approximate scalings:

$$T_i \sim \frac{\gamma^3 \sigma_s}{I_{sb}} \epsilon_{zo} \sqrt{g \langle [\delta_{acc}(s)]^2 \beta(s) \rangle} C$$  
$$T_{el} \sim \frac{\gamma^2 A_y}{P}$$  
$$T_{hs} \sim \frac{\delta_{acc}}{P}$$

with $\delta_{acc}$ the relative momentum acceptance (see Eq. (38)), $\sigma_s$ the rms bunch length, $I_{sb}$ the single bunch current, $\epsilon_{zo}$ the natural emittance from Eq. (18), $g$ the emittance ratio from Eq. (28), $P$ the residual gas pressure and $A_y$ the vertical acceptance (see Eq. (35)), assuming $A_y \ll A_x$ in any case due to the presence of narrow gap undulators in a light source lattice.

Touschek lifetime is defined as the beam half life time, since Touschek scattering is a space charge effect, i.e., a two-particle process leading to a hyperbolic beam decay. The residual gas lifetimes are defined as decay of beam current to $1/e$ of its initial value, since these effects are single particle processes leading to an exponential decay. For more information on lifetime see Refs. [17, 19, 24, 30, 31].

5.2 Injection

In order to achieve a large beam current in the storage ring, many beams ($\approx 100 \ldots 1000$) delivered by the linac/booster injection complex have to be accumulated in the storage ring. In order to keep stored particles while bringing in new particles, the scheme as sketched in Fig. 10 is applied: A closed bump of the stored beam brings it close to a current septum shielding a dipole field for injecting the injected beam from the stored beam region. Due to technical limitations the bump has a typical duration of a few microseconds corresponding to a few turns. Beyond the end of the septum stored and injected beams propagate parallel. The bump is closed for the stored beam, whereas the injected beam performs a betatron oscillation around the stored beam orbit. This oscillation continues until radiation damping merges the injected beam into the stored beam. With damping times in the order of milliseconds, this takes some 1000 turns. During this time the injection beam requires a certain amount of horizontal acceptance as shown by the large ellipses in Fig. 10.
5.3 Acceptance definition

We distinguish physical acceptances, determined by the beam pipe diameters, and dynamic acceptances, defined by the onset of chaotic or unstable motion and particle loss beyond some critical amplitude due to nonlinear resonances. Generally acceptance is defined by the 6-dimensional phase space volume where particles are stable, i.e., where they perform bounded oscillations. In most machines the coupling between the subspaces is not too strong so we may separate horizontal, vertical and longitudinal acceptance as projections from 6D to 2D-spaces.

Then the acceptances are invariants of the lattice like the Courant-Snyder invariant in case of linear uncoupled betatron motion. Local projections of the transverse acceptance to real space \(\{x, y\}\) give the dynamic apertures, which are not invariants but depend on the local beta functions. The transverse acceptance is usually measured in mm·mrad, the aperture in mm.

It is unusual to mention the two-dimensional longitudinal acceptance of the RF bucket, instead its projection to the axis of relative momentum deviation \(\delta\), called RF momentum acceptance, is quoted. The corresponding lattice momentum acceptance defines the \(\delta\)-range where non-zero transverse acceptances still exist.

Determination of dynamic acceptance is not trivial, since the equations of motion are nonlinear and thus not integrable in most cases. Stability of motion has to be proven by simulation, i.e., probing points in 6D-space on stability by tracking. A test particle is considered to be stable if it survives the tracking, i.e., its amplitudes stay within some limits. Of course this depends on the number of turns to be tracked. Electrons fortunately "forget" their history due to radiation damping, thus tracking one damping time usually is enough (\(10^3 \ldots 10^4\) turns).

It is a general criterion for lattice performance, that the pure dynamic acceptance, i.e., the phase space separatrix calculated excluding the cut off beyond beampipe, should be larger than the physical acceptance. Furthermore, the dynamic acceptance should have little nonlinear distortions, otherwise...
the actual available acceptance given as the dynamic acceptance including physical limitations, will be reduced.

5.4 Physical acceptance

An initial lattice design composed from ideal quadrupoles and bending magnets has a purely linear dynamics, and the dynamic acceptance is infinitely wide. (Actually this is not exactly true, since the equations of motions had been derived assuming paraxial motion, i.e., "small" deviations from the closed orbit.)

A particle however will be lost if $|x(s)| \geq a_x(s)$ where $a_x(s)$ is the half width of the vacuum chamber. The linear betatron motion $x(s)$ of a particle is given by Eq. (7). Since we consider many turns we drop the betatron phase and find the physical acceptance as the maximum possible betatron amplitude, $A = 2J_{max}$, by identifying $x(s)$ with its limit $a_x(s)$ and taking the minimum from all locations in the lattice:

$$A_x = \min \left( \frac{(a_x(s) - |D(s)\cdot \delta|)^2}{\beta_x(s)} \right). \quad (33)$$

Since $A_x$ is an invariant of the linear betatron motion we get the local projection of the acceptance, i.e., the minimum and maximum $x$-values a particle can reach from Eq. (7):

$$x(s) = \pm \sqrt{A_x \cdot \beta_x(s)} + D(s) \cdot \delta . \quad (34)$$

Due to the absence of vertical dispersion the corresponding equations for vertical acceptance are simpler:

$$A_y = \min \left( \frac{a_y(s)^2}{\beta_y(s)} \right) , \quad y(s) = \pm \sqrt{A_y \cdot \beta_y(s)}. \quad (35)$$

5.5 Momentum acceptance

Momentum and phase acceptance as provided by the RF bucket height and length can be considered like longitudinal physical acceptances, although they are dynamic actually, because they are almost constant along the lattice and decoupled from the transverse dynamics. But momentum acceptance is also restricted by the transverse acceptance of the lattice: From Eq. (33) we also see that $A_x$ disappears for momentum deviations $|\delta| > \min(a_x(s)/|D(s)|)$, i.e., when the closed orbit hits the vacuum chamber.

Momentum acceptance of the lattice is relevant for the Touschek beam life time (see Section 5.1): The scattering events cause a sudden change in particle momentum while leaving the transverse coordinates almost constant. After the event, the two interacting particles’ vectors are given by $(\approx 0, \approx 0, \approx 0, \approx 0, \approx 0, \pm \delta, 0)$ since the scattered particles come from the beam core where the transverse coordinates are very small. Now the particle will start a betatron oscillation relative to the dispersive closed orbit. For a linear lattice the amplitude of this oscillation is given by

$$A_x = \gamma_{xo}(D_o\delta)^2 + 2\alpha_{xo}(D_o\delta)(D'_o\delta) + \beta_{xo}(D'_o\delta)^2 = \mathcal{H}_o \delta^2 \quad (36)$$

with $\alpha_{xo}, \beta_{xo}, \gamma_{xo}$, the twiss parameters at location ‘0’ where the scattering event occurred, $D_o, D'_o$ the dispersion and its slope, and $\mathcal{H}_o$ the lattice invariant from Eq. (19).

The particles will perform oscillations according to Eq. (7) resulting at another location $s$ in the maximum excursion

$$x(s) = \frac{A_x(s) \cdot \beta_x(s)}{\mathcal{H}_o \beta_x(s) + D(s)} \cdot \delta \quad (37)$$

Like in derivation of Eq. (33) we identify $x$ with $a_x$ as loss criterion and get the local momentum acceptance for location ‘0’ as

$$\delta_{acc}(s_0) = \pm \min \left( \frac{a_x(s)}{\sqrt{\mathcal{H}_o \beta_x(s) + |D(s)|}} \right) . \quad (38)$$

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Thus the momentum acceptance provided by the lattice varies for different locations. From Eq. (38) it can be easily shown that the lattice momentum acceptance at location of maximum dispersion is half of its value for the dispersion free region.

Finally, the local momentum acceptance is the minor of the values from Eq. (38) and the momentum acceptance provided by the RF system, i.e., the height of the RF bucket. The RF momentum acceptance does not vary along the lattice and is given by

$$
\delta_{\text{acc}}^{\text{RF}} = \sqrt{\frac{2U\lambda_{\text{rf}}}{\pi E\alpha C}(\cot \varphi_s + \varphi_s - \pi/2)}
$$

(39)

with \(\sin \varphi_s = U/V_{\text{rf}}\) and \(\lambda_{\text{rf}}, V_{\text{rf}}\) RF wavelength and peak voltage, and \(U\) the energy loss per turn from Eq. (26). Beam lifetime calculations finally have to integrate scattering rates and momentum acceptances over the lattice [24].

### 5.6 Dynamic acceptance

The sextupoles to be installed for chromaticity correction (see Section 4.7) have dramatic side-effects, because they apply a parabolic, i.e., a nonlinear field. Thus the equations of motion become nonlinear and usually can not be integrated any longer. As a consequence, beyond some transverse amplitude the motion will become unstable thus defining finite dynamic acceptances.

Finding a distribution of sextupoles for correcting the large chromaticity and leaving sufficient dynamic acceptance, is the most challenging task in light source design! How to attack this problem systematically is described elsewhere [1, 25]. Here we only give a brief outline of the approach:

Analysis of the sextupole Hamiltonian reveals nine terms of first order in sextupole strength that affect the equations of motion. Two of them are the chromaticities, they are independent of betatron phases, i.e., quadrupoles and sextupoles contribute additive as expressed by Eq. (31). The seven other terms are phase dependent and thus show a resonant behaviour: The kicks from single sextupoles on the particles may partially cancel due to their different phase advances, however any residual kick may be amplified more or less, depending on the machine tune. As a consequence the so-called sextupolar resonances will be excited (see Fig. 9): There are two terms driving integer resonances of type \(Q_x\), one term driving third integer resonances \(3Q_x\), two terms driving coupling resonances \(Q_x \pm 2Q_y\) and two terms driving chromatic half integer resonances \(2Q_x, 2Q_y\) for off-momentum particles.

Quadrupoles contribute to four of the nine terms: They are the source of the two chromaticities, i.e., the variation of tunes with momentum, and they also contribute to the \(2Q_x, 2Q_y\) terms, which cause variation of the betafunctions with momentum and subsequently higher order chromaticities. Clearly, the ideal distribution of sextupoles in a lattice would compensate the four chromatic quadrupole terms while maintaining a total cancellation of the five adverse sextupole terms through appropriate phase advances.

Since all nine terms are of first order (i.e., linear) in sextupole strength, they form a linear system of equations. If we have \(M\) sextupole families, a procedure like singular value decomposition (SVD) would return the \(M\)-vector of sextupole strengths for the optimum sextupole pattern. Basically, for \(M \geq 9\) a solution should exist that corrects all chromatic effects while not exciting any resonances.

In practise however, in particular in low emittance light sources, the linear system tends to degenerate, because the horizontal betatron phase advance per cell is rather close to \(180^\circ\) due to strong horizontal focusing for achieving low emittance (see Fig. 7), thus the individual sextupole contributions to the phase dependant \(2Q_x\) term add up coherently, and no solution will be found for the sextupole pattern. Instead appropriate phase advances have to be inserted into the machine, mainly by exploiting the straight sections of the storage ring for that purpose, to create a sextupole pattern (in terms of betatron phases) which shows less degeneracy and may have a chance to find a set of strengths that works. Clearly, manipulating the straights for the sake of phase advances has a large impact on the general machine layout and performance. As a consequence, the following guideline emerges: for lattices with large
chromaticites linear and nonlinear lattice design are not decoupled. One must not proceed by designing the bending magnet and quadrupole arrangement first and add the sextupoles later. Instead from the early planning of the lattice the sextupole pattern has to be taken into account.

Another complication arises from the crosstalk between sextupoles causing higher order effects. If the sextupoles are rather strong, as it is the case in a low emittance light source, also the second order sextupole Hamiltonian has to be considered. It consists of 13 terms causing tune shifts with amplitude, second order chromaticities and excitation of octupolar resonances. Any light source design has to consider at least the amplitude dependant tune shifts in order to achieve sufficient dynamic acceptance.

In practice, a minimization procedure will vary the vector of sextupole family strengths in order to suppress a penalty function constructed by application of suitable weight factors to first and second order terms. Results achieved during this procedure have to be checked by tracking calculations.

Figure 11 gives an example of successful dynamic acceptance recovery through minimization of the 9 first order and 13 second order sextupole terms by means of 9 sextupole families.

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