ON THE EFFECTS OF SKEW QUADRUPOLES AND TILTED MAGNETS

H.G. Hereward, E. Keil, and J.D. Young

SUMMARY

In this report some analytical and computational results are given, describing the effects of tilted magnets and of their correction by skew quadrupoles on the betatron motion, the momentum compaction function \( \alpha_p(s) \), and on the closed orbit.

In analogy to the usual formalism using pairs of \( 3 \times 3 \) matrices for the study of the uncoupled particle motion we employ a \( 5 \times 5 \) matrix formalism. The matrices thus obtained are analysed in terms of the angles for the principal planes for the betatron motion, and in terms of eigenvectors for the momentum compaction function and the closed orbit.

The results obtained show that the betatron motion can be made reasonably uncoupled by a limited number of skew quadrupoles distributed around the circumference. The effects of the tilted magnets and skew quadrupoles together on the momentum compaction function and on the closed orbit can be kept within tolerable limits. The gradients and distribution of skew quadrupoles necessary to achieve this are given.

Some details about the computer programs used are given in an Appendix.
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1. **INTRODUCTION**

In the ISR, coupling effects due to tilted magnets are particularly disturbing because the stacking of many pulses from the CPS produces a rather wide flat beam. For this reason, a certain number of skew quadrupoles has been foreseen in their design. However, it was felt that some justification of the choice of their number should be given. Also, it seemed interesting to know how well the coupling could be removed by using skew quadrupoles and what gradients and tolerances are involved in order to achieve this. At the same time possible effects of skew quadrupoles apart from the decoupling effect should be looked into.

In this report we first present the arithmetic chosen for treating the effects of skew quadrupoles and tilted magnets. We then apply it to a rather systematic study of the effects that they can have on the particle motion. The information collected is finally used to guide the choice of the number of skew quadrupoles, and their strength.

2. **A MATRIX FORMALISM FOR TILTED MAGNETS AND SKEW QUADRUPOLES**

For the present discussion we define two coordinate systems, the (R,Z) - system, and the (U,V) - system.

The (R,Z) - system is common to all magnets in the structure. At every azimuth it has its origin at the ideal equilibrium orbit, the R axis is in the median plane and points away from the machine centre, the Z axis is perpendicular to the median plane. The (R,Z) - system is used for all operations involving several elements. Therefore all matrices describing individual elements should be expressed in this system.

The (U,V) - system, on the other hand, refers to individual magnets. Its origin is on the equilibrium orbit; the U axis is in the median plane of the magnetic field of that element; the V axis is perpendicular to it. The advantage of this system is that there we know the transformation matrices for the magnets. The problem of finding the transformation matrices in the (R,Z) - system is thus reduced to finding the transformation between the (R,Z) and the (U,V) - systems which is just a rotation by the tilt angle \( \phi \).

In an untitled magnet the R and the U axes, and the Z and the V axes, coincide.
2.1 Matrices for individual elements

2.1.1 Tilted magnet matrix, momentum compaction mode

In the \((U,V)\)-system we just have normal uncoupled magnet matrices. If we limit ourselves to magnets with dispersion in the U direction only, as will usually be the case, we can write these matrices down symbolically, the index \(o\) referring to the beginning of a magnet, the index \(\ell\) referring to its end:

\[
\begin{pmatrix}
U_1 \\
U_1' \\
\Delta p/p
\end{pmatrix}
=
\begin{pmatrix}
a_{11} & a_{12} & a_{15} \\
a_{21} & a_{22} & a_{25} \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
U_0 \\
U_0' \\
\Delta p/p
\end{pmatrix}
\]  \hspace{1cm} (1)

\[
\begin{pmatrix}
V_1 \\
V_1' \\
\Delta p/p
\end{pmatrix}
=
\begin{pmatrix}
b_{11} & b_{12} & 0 \\
b_{21} & b_{22} & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
V_0 \\
V_0' \\
\Delta p/p
\end{pmatrix}
\]  \hspace{1cm} (2)

This pair of \(3\times3\) matrices is equivalent to the following \(5\times5\) matrix:

\[
\begin{pmatrix}
U_1 \\
U_1' \\
V_1 \\
V_1' \\
\Delta p/p
\end{pmatrix}
=
\begin{pmatrix}
a_{11} & a_{12} & 0 & 0 & a_{15} \\
a_{21} & a_{22} & 0 & 0 & a_{25} \\
0 & 0 & b_{11} & b_{12} & 0 \\
0 & 0 & b_{21} & b_{22} & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
U_0 \\
U_0' \\
V_0 \\
V_0' \\
\Delta p/p
\end{pmatrix}
\]  \hspace{1cm} (3)

In order to obtain the magnet matrix in the \((R,Z)\)-system we must transform the initial conditions into the \((U,V)\)-system and then back into the \((R,Z)\)-system. We know that these transformations are straightforward rotations by the tilt angle \(\varphi\). They are in \(5\times5\) matrix form:
\[ \begin{pmatrix} U & V & V' & \Delta p \\ \cos \varphi & 0 & \sin \varphi & 0 & 0 \\ 0 & \cos \varphi & 0 & \sin \varphi & 0 \\ -\sin \varphi & 0 & \cos \varphi & 0 & 0 \\ 0 & -\sin \varphi & 0 & \cos \varphi & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} R & R' \\ 0 & \cos \varphi & 0 & \sin \varphi & 0 \\ 0 & 0 & \cos \varphi & 0 & \sin \varphi \\ 0 & 0 & 0 & \cos \varphi & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \]

The inverse transformation is obtained by changing the sign of \( \varphi \). Combining all transformations we arrive at the following product of three matrices describing the effect of a tilted magnet, Eq. (5).

\[ \begin{pmatrix} L & L' \\ \cos \varphi & 0 & -\sin \varphi & 0 & 0 \\ 0 & \cos \varphi & 0 & -\sin \varphi & 0 \\ \sin \varphi & 0 & \cos \varphi & 0 & 0 \\ 0 & \sin \varphi & 0 & \cos \varphi & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} L & L' \\ \cos \varphi & 0 & \sin \varphi & 0 & 0 \\ 0 & \cos \varphi & 0 & \sin \varphi & 0 \\ -\sin \varphi & 0 & \cos \varphi & 0 & 0 \\ 0 & -\sin \varphi & 0 & \cos \varphi & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \]

Performing the matrix multiplication we have Eq. (6).
Specifically, we have for F magnets, D magnets, straight sections, skew F quadrupoles, skew D quadrupoles the matrices given in Eqs. (9a), (9b), (10), (11a), (11b). We use the following notation: \( L \) is the length of a magnet, \( \rho \) its radius of curvature. Its strength is defined by \( K = \frac{1}{B_p} \frac{dB_y}{du} \) where \( B_p = p/e \) is the magnetic rigidity of the particles. The sign convention is \( K > 0 \) for F magnets. We use the following abbreviations:

\[
\begin{align*}
K_u &= |K - \frac{1}{\rho^2}| & K_v &= |K| \\
\psi_u &= \sqrt{K_u} L & \psi_v &= \sqrt{K_v} L
\end{align*}
\]

(7)

The reciprocal focal length \( C \) of a magnet becomes in short lens approximation:

\[
C = KL
\]

(8)

Skew quadrupoles have \( \phi = \pi/4 \) and \( \rho = \infty \), thus \( K_u = K_v = K \) and \( \psi_u = \psi_v = \psi \).

Straight section matrix:

\[
\begin{pmatrix}
1 & L & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & L & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

(10)
\( \text{F magnet matrix:} \)

\[
\begin{pmatrix}
\cos \theta_u \cos \phi + \cosh \theta_u \sin \phi \\
\sin \theta_u \cos \phi + \sinh \theta_u \sin \phi \\
- \sqrt{K_u} \sin \theta_u \cos \phi - \sqrt{K_u} \sinh \theta_u \sin \phi \\
\cos \theta_u \cos \phi - \cosh \theta_u \sin \phi \\
\sin \theta_u \cos \phi - \sinh \theta_u \sin \phi \\
- \sqrt{K_u} \sin \theta_u \cos \phi + \sqrt{K_u} \sinh \theta_u \sin \phi \\
0 \\
0 \\
0 \\
1
\end{pmatrix}
\]

\( \text{D magnet matrix:} \)

\[
\begin{pmatrix}
\cosh \theta_u \cos \phi + \cosh \theta_u \sin \phi \\
\sinh \theta_u \cos \phi + \sinh \theta_u \sin \phi \\
\sqrt{K_u} \sinh \theta_u \cos \phi - \sqrt{K_u} \sinh \theta_u \sin \phi \\
\cosh \theta_u \cos \phi - \cosh \theta_u \sin \phi \\
\sinh \theta_u \cos \phi - \sinh \theta_u \sin \phi \\
\sqrt{K_u} \sinh \theta_u \cos \phi + \sqrt{K_u} \sinh \theta_u \sin \phi \\
\sin \theta_u \cos \phi + \sin \theta_u \sin \phi \\
\sin \theta_u \cos \phi - \sin \theta_u \sin \phi \\
\cosh \theta_u \cos \phi - \cosh \theta_u \sin \phi \\
\sinh \theta_u \cos \phi - \sinh \theta_u \sin \phi \\
0 \\
0 \\
0 \\
1
\end{pmatrix}
\]
2.1.2 Tilted magnet matrix, closed orbit mode

The closed orbit due to field errors (and misalignments) may also be found using a 5×5 matrix formalism if the matrices operate on column vectors of the form

\[
\begin{pmatrix}
R \\
R' \\
Z \\
Z' \\
1
\end{pmatrix}
\]
and if the fifth columns of the matrices contain elements describing the change in $R$, $R'$, $Z$ and $Z'$ due to the field error (or misalignment).

The field errors introduced by a magnet tilt are in the $R$ direction:

$$\frac{\Delta B}{B_v} = \sin \varphi,$$

and in the $Z$ direction:

$$\frac{\Delta B}{B_v} = \cos \varphi - 1.$$

Therefore we write for the closed orbit mode matrix of a tilted magnet:

$$\begin{bmatrix}
R_x' \\
R'_x \\
Z_x' \\
Z'_x \\
1
\end{bmatrix} =
\begin{bmatrix}
B_{xx} & B_{yx} & B_{zx} & B_{xz} & B_{x5} \\
B_{y4} & B_{yy} & B_{zy} & B_{zy} & B_{y5} \\
B_{z4} & B_{zy} & B_{zz} & B_{zz} & B_{z5} \\
B_{z5} & B_{z5} & B_{z5} & B_{z5} & B_{z5} \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
R_x \\
R'_x \\
Z_x \\
Z'_x \\
1
\end{bmatrix}$$

(15)

2.2 **Analysis of the matrix for a superperiod**

The matrix describing an entire superperiod is the product of the matrices of the individual elements forming the superperiod, obtained by the usual rules of matrix multiplication. We analyse it separately whether we are concerned about the coupled betatron motion, the momentum compaction function, or the closed orbit.

2.2.1 **Analysis of the coupled betatron motion**

To investigate the betatron oscillations we do not require the fifth row or column of the matrices, and can simply suppress them. All matrices in this section are $4 \times 4$. 
We have seen that the rotation of a magnet by a tilt angle $\varphi$ changes the four radial matrix elements $a_{11}$, etc., and the four vertical matrix elements $b_{11}$, etc., only to second order in $\varphi$, while introducing a first order change in the eight zero elements of the untitled case. It can be shown, in fact, that the matrix for any magnet structure, including tilted elements and skew quadrupoles, can always be written in the form:

$$S = S_0 + S_1 + \text{second and higher order terms,} \quad (16)$$

where $S_0$ is the matrix with no skew fields, of form:

$$S_0 = \begin{pmatrix}
  a_{11} & a_{12} & 0 & 0 \\
  a_{21} & a_{22} & 0 & 0 \\
  0 & 0 & b_{11} & b_{12} \\
  0 & 0 & b_{21} & b_{22}
\end{pmatrix} \quad (17)$$

and $S_1$ is of first order in the skew fields (whether from tilts or from skew quadrupoles) and has the form:

$$S_1 = \begin{pmatrix}
  0 & 0 & c_{11} & c_{12} \\
  0 & 0 & c_{21} & c_{22} \\
  d_{11} & d_{12} & 0 & 0 \\
  d_{21} & d_{22} & 0 & 0
\end{pmatrix} \quad (18)$$

Since we expect to be dealing with magnet tilts in the region of a few milliradians, and with skew quadrupole strengths appropriate for correcting such tilts, it should be sufficiently accurate to work only to first order in the skew fields. This considerably simplifies matters.
Further simplification is introduced if we limit ourselves to structures which have backwards/forwards symmetry, and always calculate matrices for a superperiod which begins and ends at symmetry points. For the unperturbed matrix $S_0$ we then know

$$
\begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{pmatrix}
= 
\begin{pmatrix}
  \cos \mu_R & \beta_R \sin \mu_R \\
  -\frac{1}{\beta_R} \sin \mu_R & \cos \mu_R
\end{pmatrix}
$$

(19)

and correspondingly, for the four $Z$ elements $b_{11}$, etc. Here the $\mu$'s are phase shifts for one superperiod and these $\beta$'s are for the chosen symmetry points.

Imposing the same symmetry on the superperiod with its skew perturbations requires that

$$S \bar{S} = 1$$

(20)

where we use the bar above the matrix symbol to indicate that half the signs have been changed, according to the scheme:

$$
\begin{pmatrix}
  + & - & + & - \\
  - & + & - & + \\
  + & - & + & - \\
  - & + & - & +
\end{pmatrix}
$$

(21)

Substituting Eq. (16) into Eq. (20) and taking first-order terms we find

$$S_0 \bar{S}_1 + S_1 \bar{S}_0 = 0 .$$

(22)

Multiplying out the left-hand side there are eight elements that are not identically zero, so eight equations restrict the elements $c_{11}$, etc., and $d_{11}$, etc., but they are not all independent, and reduce to the two pairs
\[
c_{11} = \left( -c_{12} \frac{\sin \mu_Z}{\beta_Z} + c_{21} \beta_R \sin \mu_R \right) / \left( \cos \mu_Z + \cos \mu_R \right)
\]
\[
c_{22} = \left( -c_{12} \frac{\sin \mu_R}{\beta_R} + c_{21} \beta_Z \sin \mu_Z \right) / \left( \cos \mu_Z + \cos \mu_R \right)
\]
\[
d_{11} = \left( -d_{12} \frac{\sin \mu_R}{\beta_R} + d_{21} \beta_Z \sin \mu_Z \right) / \left( \cos \mu_R + \cos \mu_Z \right)
\]
\[
d_{22} = \left( -d_{12} \frac{\sin \mu_Z}{\beta_Z} + d_{21} \beta_R \sin \mu_R \right) / \left( \cos \mu_R + \cos \mu_Z \right)
\]

so that \( S \) is determined as soon as the four elements on its cross-diagonal are given.

It is also known that the matrix \( S \) must be symplectic [see Courant and Snyder, Annals of Physics 2, 30 (1958)] and, given Eqs. (23), one finds that this imposes the further condition on the elements of \( S \):

\[
\begin{align*}
d_{11} & = c_{22} \\
d_{12} & = c_{12} \\
d_{21} & = c_{21} \\
d_{22} & = c_{11}
\end{align*}
\]

so that finally \( S \) can be written in terms of two arbitrary elements only, \( c_{12} \) and \( c_{21} \). For shortness we have put \( C = (\cos \mu_Z + \cos \mu_R)^{-1} \).

\[
\begin{pmatrix}
0 & 0 & (-c_{12} \frac{\sin \mu_Z}{\beta_Z} + c_{21} \beta_R \sin \mu_R) C & c_{11} \\
0 & 0 & c_{22} & (-c_{12} \frac{\sin \mu_R}{\beta_R} + c_{21} \beta_Z \sin \mu_Z) C \\
(-c_{12} \frac{\sin \mu_Z}{\beta_Z} + c_{21} \beta_R \sin \mu_R) C & c_{21} & 0 & 0 \\
(-c_{12} \frac{\sin \mu_R}{\beta_R} + c_{21} \beta_Z \sin \mu_Z) C & c_{22} & 0 & 0
\end{pmatrix}
\]
Guided by our knowledge of the behaviour of a pair of linear harmonic oscillators with a conservative coupling we shall interpret the matrix $S_0 + S_1$ by finding a change of variables which will transform it into the uncoupled form $S_0$. As symbols for the new variables we shall use $U, U', V, V'$, but we must give them greater liberty than that of the simple rotation of coordinates [Eq. (4)].

Since we are treating the skew gradients only in first approximation we can also suppose that $U$ and $U'$ are nearly radial and $V$ and $V'$ nearly vertical. We put

\[
\begin{align*}
R &= U - \alpha V \\
Z &= \gamma U + V
\end{align*}
\]  

(26)

where $\alpha$ and $\gamma$ are small. If we find a change of variables such that $U$ and $V$ are uncoupled, we shall have one mode of betatron oscillations with $V$ always zero, so that the particle always falls on the nearly-horizontal line

\[ Z = \gamma R \]  

(27)

and another mode with $U$ zero, so of nearly vertical polarization

\[ R = -\alpha Z \]  

(28)

For the moment we have no reason to suppose that $\alpha$ and $\gamma$ must be equal.

We shall regard $\alpha$ and $\gamma$, and indeed $U, U', V, V'$, as being defined only at the junctions between each superperiod and the next. Thus a pure $U$-mode oscillation means that the particle falls on the line [Eq. (27)], at every successive junction, but says nothing about where it goes within the superperiod.

We shall also put

\[
\begin{align*}
R' &= U' - \alpha' V' \\
Z' &= \gamma' U' + V'.
\end{align*}
\]  

(29)
The primes on \( \alpha' \) and \( \gamma' \) mean only that they are not necessarily the same as \( \alpha \) and \( \gamma \). One would rather expect them to be different because of \( \beta_R \neq \beta_Z \) for the unperturbed structure.

![Diagram of angles of polarization at the superperiod ends](image)

**Figure 1**

Angles of polarization at the superperiod ends

Thus for our change of variables we have

\[
\begin{pmatrix}
R \\
R' \\
Z \\
Z'
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & -\alpha & 0 \\
0 & 1 & 0 & -\alpha' \\
\gamma & 0 & 1 & 0 \\
0 & \gamma' & 0 & 1
\end{pmatrix}
\begin{pmatrix}
U \\
U' \\
V \\
V'
\end{pmatrix}
\] (30)
This matrix we shall call \( C \). To first order in the \( \alpha \)'s and \( \gamma \)'s its reciprocal can be obtained by simply changing their signs.

If the matrix of a superperiod for the variables \( U \ldots \) has the uncoupled form \( S_0 \), then for the variables \( Z \ldots \) the matrix is

\[
C S_0 C^{-1}
\]

and we look for a \( C \) that makes this equal to

\[
S_0 + S_1 + \text{second and higher order terms}
\]

where \( S_1 \) has the form given in Eq. (25).

Evaluating \( C S_0 C^{-1} - S_0 \) to first order one finds that it automatically satisfies the backward/forward symmetry conditions [Eq. (23)], but it is consistent with symplecticity, [Eq. (24)], if and only if one has

\[
\begin{align*}
\alpha' &= \gamma \\
\gamma' &= \alpha
\end{align*}
\]

Then we get

\[
S_1 =
\begin{pmatrix}
0 & 0 & \gamma (\cos \mu_R - \cos \mu_Z) & \gamma \beta_R \sin \mu_R - \alpha \beta_Z \sin \mu_Z \\
0 & 0 & \gamma \beta_R \sin \mu_R - \alpha \beta_Z \sin \mu_Z & 0 \\
\gamma (\cos \mu_R - \cos \mu_Z) & \gamma \beta_R \sin \mu_R - \alpha \beta_Z \sin \mu_Z & 0 & 0 \\
\gamma \beta_R \sin \mu_R - \alpha \beta_Z \sin \mu_Z & \gamma (\cos \mu_R - \cos \mu_Z) & 0 & 0
\end{pmatrix}
\]

(33)
and in general *) $\alpha$ and $\gamma$ can be chosen uniquely to make this agree with any computed $S_t$ of the form [Eq. (25)], for example by taking

$$
\alpha = c_{11}/(\cos \mu_R - \cos \mu_Z) \\
\gamma = c_{22}/(\cos \mu_R - \cos \mu_Z).
$$

(34)

### 2.2.2 Computation of the momentum compaction function

Let $S(s)$ be the matrix for a superperiod starting at the azimuthal position $s$. The closed orbit at this azimuth for a particle with a momentum error $\Delta p/p$ is then the solution of the equation:

$$
\begin{pmatrix}
R(s) \\
R'(s) \\
Z(s) \\
Z'(s) \\
\Delta p/p
\end{pmatrix} = S(s) \begin{pmatrix}
R(s) \\
R'(s) \\
Z(s) \\
Z'(s) \\
\Delta p/p
\end{pmatrix}
$$

This equation can be considered as an inhomogeneous equation in the four unknowns $R, R', Z, Z'$ with right-hand sides given by the fifth column of the matrix $S$ multiplied by $(- \Delta p/p)$; it can be solved by inverting the $4 \times 4$ top left submatrix of $S$. The two-component vector

$$
\alpha_p(s) = [R(s), Z(s)]/(\Delta p/p)
$$

is called the momentum compaction function. In an uncoupled machine its $Z$ component is zero and $\alpha_p(s)$ is considered as a scalar. The changes due to the

*) If $\cos \mu_1 - \cos \mu_2 = 0$ our formulae give infinite $\alpha$ and $\gamma$. This is because in such a case $\alpha$ and $\gamma$ are substantial even with small coupling, and we worked to first order in $\alpha$ and $\gamma$ so our formulae fail if they are not small.
tilt of the magnets are of second order for the R component, and of first order for the Z component. Therefore we are mainly concerned about the Z component.

2.2.3 Computation of the distorted closed orbit

The same analysis as given in section 2.2.2 for the computation of the momentum compaction function applies to the distorted closed orbit, with the momentum compaction mode matrices replaced by the closed orbit mode ones.

3. SYSTEMATIC ANALYSIS OF THE EFFECTS OF TILTED MAGNETS AND SKEW QUADRUPOLES

As should have become clear from the preceding discussion we may expect the following three effects of tilted magnets and skew quadrupoles:

i) a coupling of the horizontal and vertical betatron motion;

ii) a change in the momentum compaction function, in particular the existence of a vertical component;

iii) a distortion of the closed orbits, mainly of the vertical one.

We feel that most of the essentially different arrangements of tilted magnets and skew quadrupoles are covered when we investigate the following three only:

i) systematically tilted magnets, either with equal or with opposite sign of the tilt in F and D magnets;

ii) randomly tilted magnets with tilt angles picked from a normal distribution with an r.m.s. tilt $\tilde{\phi}_{r.m.s.}$;

iii) skew quadrupoles in predetermined arrangements.

The systematic analysis to be presented in this chapter is thus logically divided into discussions of each combination of arrangements and effects.

The relevant orbit parameter for the machine used in this study are listed in Table 1.
Table 1

Orbit parameters

<table>
<thead>
<tr>
<th></th>
<th>Horizontal</th>
<th>Vertical</th>
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</thead>
<tbody>
<tr>
<td>Q - value</td>
<td>8.692</td>
<td>8.800</td>
</tr>
<tr>
<td>Phase advance / superperiod</td>
<td>2.173 x 2\pi</td>
<td>2.200 x 2\pi</td>
</tr>
<tr>
<td>(\beta) value at the symmetry point</td>
<td>19.7</td>
<td>12.7</td>
</tr>
<tr>
<td>(\alpha_p) value at the symmetry point</td>
<td>2.205</td>
<td>0</td>
</tr>
</tbody>
</table>

3.1 Action of systematically tilted magnets

Table 2 contains a list of amplification factors for the matrix elements \(c_1 = d_{22}, c_{22} = d_1\), the maximum vertical component of the momentum compaction function \(\alpha_{pv \, max}\) and of the closed orbit \(Z_{max}\). The actual quantity is obtained from these amplification factors by multiplying them with the tilt angles \(\varphi\). We checked that they are linear in \(\varphi\) to a very good approximation.

The results are for an ISR structure with gradients in the F and D magnets such that the Q-values given in Table 1 are obtained. The matrix elements are those for the centre of the inner arc, which is a symmetry point as required for their analysis in terms of the angles between the principal planes and the R and Z axes also included in the table. The equalities for matrix elements given above are also due to the symmetry.

Table 2 shows that the drastic coupling and momentum compaction function effects are due to tilts with opposite sign for F and D magnets, the effects of magnets with equal tilts are very much smaller. In fact, the amplification factors for the angles of the principal planes in this case are just unity, which indicates that the betatron motions are uncoupled in a coordinate system rotated with the magnets, as one would expect.
Table 2.

Amplification factors for systematically tilted magnets

<table>
<thead>
<tr>
<th>Tilt</th>
<th>Equal sign</th>
<th>Opposite sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_{11} = d_{22}$</td>
<td>0.155605</td>
<td>100.35 rad$^{-1}$</td>
</tr>
<tr>
<td>$c_{22} = d_{11}$</td>
<td>0.155605</td>
<td>67.97 rad$^{-1}$</td>
</tr>
<tr>
<td>$</td>
<td>\alpha_{\text{vmax}}</td>
<td>$</td>
</tr>
<tr>
<td>$</td>
<td>z_{\text{inax}}</td>
<td>$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>1.00</td>
<td>667</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>1.00</td>
<td>452</td>
</tr>
</tbody>
</table>

The only effect which is more pronounced for equally tilted magnets is that on the vertical closed orbit. This is to be expected since in this case all the vertical kicks due to the tilt add, whereas for opposite tilt they are of opposite sign in F and D magnets and thus cancel to a large extent.

The general case, where all F magnets have a certain tilt angle and all D magnets have another tilt angle $\varphi_D$, may be considered as the sum of two effects: a systematic tilt of all magnets with equal sign corresponding to the average $(\varphi_F + \varphi_D)/2$, and a systematic tilt with opposite sign in F and D magnets corresponding to half the difference $(\varphi_F - \varphi_D)/2$.

The above analysis then shows that the betatron coupling and the vertical momentum compaction are essentially due to the difference of the F and D tilt angles, and the closed orbit distortion is due to their sum.
3.2 Action of randomly tilted magnets

Table 3 gives a list of the r.m.s. amplification factors for the matrix elements $a_{11}$, $c_{22}$, $d_{41}$, $d_{22}$ at the centre of the inner arc, and the vertical components of the maximum momentum compaction function and the vertical closed orbit distortion; the averages are taken over a sample of 20 machines. Since now the centre of the inner arc is no longer a symmetry point for the magnet tilt, the pairs of matrix elements of individual machines in the sample are no longer equal. Their r.m.s. values are only approximately so, although one would expect them to be for a large enough sample.

**Table 3**

Root mean square amplification factors for random tilts

<table>
<thead>
<tr>
<th></th>
<th>40.4</th>
<th>rad(^{-1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_{11}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d_{22}$</td>
<td>43.4</td>
<td>rad(^{-1})</td>
</tr>
<tr>
<td>$c_{22}$</td>
<td>27.2</td>
<td>rad(^{-1})</td>
</tr>
<tr>
<td>$d_{41}$</td>
<td>24.5</td>
<td>rad(^{-1})</td>
</tr>
<tr>
<td>$</td>
<td>\alpha_{p\text{vmax}}</td>
<td>$</td>
</tr>
<tr>
<td>$</td>
<td>Z_{\text{max}}</td>
<td>$</td>
</tr>
</tbody>
</table>

Comparing the figures of Tables 2 and 3 shows that the randomly tilted magnets introduce coupling which is smaller than that of systematically tilted magnets by a factor in excess of two, but that their effects on the vertical momentum compaction function and on the closed orbit is considerably bigger.
3.3 Action of skew quadrupoles

For skew quadrupoles considerably more degrees of freedom are available than for tilted magnets where the only free parameter is the angle of tilt, all others being given by other considerations. The number, arrangement and strength of skew quadrupoles can be chosen freely, within certain limits.

The analysis presented in section 2.2.1 shows that in general two independently adjustable quantities are necessary and sufficient to make $S_z$ identically zero and so decouple the betatron motions at the superperiod ends. To do this with skew quadrupoles which preserve the symmetry and superperiodicity requires, apart from special cases, two symmetrical pairs of skew quadrupoles in each superperiod, so a total of 16 per ring.

Skew quadrupoles have no effect on the closed orbit since their field vanishes at the equilibrium orbit. However, they have an effect on the vertical momentum compaction function. The total amount of skew fields is given by the tilt of the magnets which one wants to correct. Hence, one may expect that the effect of all skew quadrupoles together on the vertical momentum compaction function is approximately proportional to the inverse square root of their number, assuming that their individual contributions add in a random fashion.

Thus, for decoupling the betatron motions 16 skew quadrupoles are sufficient, whereas a larger number of them is desirable to keep their effect on the momentum compaction function small.

Figure 2 shows arrangements of various numbers of skew quadrupoles that were computed. Table 4 summarizes the results. Two cases are included in Table 4:

1) The magnets are tilted. The signs of the tilt angle are different in F and D magnets. The skew quadrupoles are excited such that the betatron motions are uncoupled within computing accuracy in the middle of the inner arcs. This case is the realistic one, which will occur in practice.

ii) The pure skew quadrupole effect is given with their gradients equal to those of case i), and magnets not tilted.
In the arrangements I to VI every N\textsuperscript{th} F-D straight section in each half superperiod is replaced by a skew quadrupole, where N = 1, 2, 3. Alternate skew quadrupoles have equal gradients. These arrangements are not possible, because they either occupy too many straight sections, or some straight sections allocated to different pieces of equipment; this limitation is just disregarded. Arrangement I has 96, II and III have 48, and IV, V and VI have 32 skew quadrupoles per ring.

Arrangements VII to IX are possible with the present layout of the ISR if a total number of four Terwilliger quadrupoles in each ring is replaced by skew ones; all these arrangements have 32 skew quadrupoles, the difference between them is how they are connected to the power supplies.

Arrangement X contains a total number of 24 skew quadrupoles, connected to three power supplies. The additional degree of freedom was used to minimize $a_{p\nu \text{max}}$, but this was not very successful.

Arrangements XI to XIII use 16 skew quadrupoles only in every ring, the minimum number required for decoupling the betatron motions. Arrangements X to XIII are also possible in the present layout of the ISR.

Table 4 requires a few comments. The trend mentioned before that the effect of skew quadrupoles on the momentum compaction function should become smaller with an increasing number of them clearly shows up. Comparing the momentum compaction functions with skew quadrupoles to that with tilted magnets alone shows that the former is usually bigger than the latter. In other words, by decoupling the betatron motions with localized skew quadrupoles, one deteriorates $a_{p\nu \text{max}}$. For a given number of skew quadrupoles there can be large differences in their gradients and in their effect on the momentum compaction function, depending on their distribution. The worst effect on $a_{p\nu \text{max}}$ always occurs in those arrangements where the gradients of the two groups of quadrupoles also are very different. This suggests that there are certain rules for their disposition which should be respected in order to achieve a minimum perturbation of $a_{p\nu \text{max}}$. A comparison between good and bad cases shows that the phase shift between unlike skew quadrupoles should be in the vicinity of $\frac{n \pi}{2}$, and that it should not be in the neighbourhood of $n\pi$, where n is an integer.
| Arrangement | Number of Q | $I_Q$ [m] | $K_1$ [m$^{-2}$/rad] | $K_2$ [m$^{-2}$/rad] | $K_3$ [m$^{-2}$/rad] | $|\alpha_{p_{\text{pvmax}}}|$ [m/rad] | Run | $|\alpha_{p_{\text{pvmax}}}|$ [m/rad] | Run |
|-------------|-------------|-----------|----------------|----------------|----------------|-------------------|---|-------------------|---|
| I           | 96          | 1.567     | 0.2664         | 0.2240         | -               | 31.685            | 10103 | 2.015             | 10066 |
| II          | 48          | 1.567     | 0.4578         | 0.4870         | -               | 46.700            | 10101 | 20.680            | 10062 |
| III         | 48          | 1.567     | 0.5254         | 0.4912         | -               | 52.505            | 10102 | 21.870            | 10064 |
| IV          | 32          | 1.567     | 0.9256         | 0.5546         | -               | 61.710            | 10098 | 36.190            | 10054 |
| V           | 32          | 1.567     | 0.5880         | 0.7566         | -               | 62.915            | 10099 | 34.215            | 10056 |
| VI          | 32          | 1.567     | 2.9875         | -1.2758        | -               | 255.540           | 10100 | 250.985           | 10058 |
| VII         | 32          | 0.4       | 2.4061         | 2.9430         | -               | 59.500            | 10097 | 29.290            | 10050 |
| VIII        | 32          | 0.4       | 3.8918         | 1.9578         | -               | 128.360           | 10097 | 112.860           | 10050 |
| IX          | 32          | 0.4       | 3.5172         | 2.0773         | -               | 64.765            | 10097 | 38.600            | 10050 |
| X           | 24          | 0.4       | -10.0          | 6.105          | 8.710           | 196.570           | 10105 | 156.080           | 10072 |
| XI          | 16          | 0.4       | 42.2372        | -10.1493       | -               | 1547.395          | 10104 | 1376.325          | 10070 |
| XII         | 16          | 0.4       | 3.1108         | 6.8510         | -               | 253.330           | 10104 | 218.380           | 10070 |
| XIII        | 16          | 0.4       | 10.4025        | 5.2253         | -               | 477.570           | 10104 | 429.955           | 10070 |
One has to accept the fact that any correction of the tilt of the magnets which is not as distributed as the magnets are, will have a worse effect than the magnets themselves. However, a comparison of the values of $|\alpha_{pv\text{ max}}|$ given in Table 4 shows that the effects of the magnets and of their correction on the momentum compaction are of opposite sign, such that $|\alpha_{pv\text{ max}}|$ is, in general, smaller for the corrected structure than for the one without tilt in the magnets but the same strength of the skew quadrupoles.

Although two independent skew quadrupole gradients are required to achieve decoupling of the betatron motions within computer accuracy, a very substantial reduction of the angles between the principal planes and the R and Z axes can be obtained with the same gradient in all skew quadrupoles. This is shown in Table 5, which contains the optimum gradient and the optimum angles $\alpha$ and $\gamma$ for various skew quadrupole arrangements. In this context the optimum is taken to be the minimum of $|\alpha| + |\gamma|$. Since $\alpha$ depends more strongly on the skew quadrupole gradient than $\gamma$, this minimum always occurs at the zeros of $\alpha$. The residual values of $\gamma$ are between one and two orders of magnitude smaller than the uncorrected ones. It thus appears that the perfect decoupling with two different skew quadrupole gradients and the approximate one with one skew quadrupole gradient only are experimentally hardly distinguishable.

A comparison of Tables 4 and 5 shows that the perturbation of the momentum compaction function tends to be smaller when the skew quadrupoles are all equal, sometimes by substantial factors. In those cases where it is bigger the difference is not very large. It might, therefore, be preferable in some cases to use equal gradients in all skew quadrupoles and to tolerate a small amount of residual coupling.

4. **CONCLUSIONS**

4.1 **Tolerances of magnet tilts without correction**

Table 6 summarizes the important effects of tilted magnets, both for systematic and random tilts. For comparison we give the angle of the principal planes also for randomly tilted magnets although our analysis is not strictly applicable to this case.
Table 5

Compensation of magnet tilt with equal skew quadrupoles

| Arrangement | Number of Q | L_Q [m] | K [m^-2/rad] | \(|a_{pv\ max}| [m/rad] | | | Run |
|-------------|-------------|---------|--------------|---------------------|---|---|---|
| I           | 96          | 1.567   | 0.244        | 0.635               | 0 | 0.1 | 10095 |
| II          | 48          | 1.567   | 0.474        | 19.075              | 0 | 0.9 | 10093 |
| III         | 48          | 1.567   | 0.506        | 19.545              | 0 | 0.8 | 10094 |
| IV          | 32          | 1.567   | 0.729        | 24.360              | 0 | 16.7 | 10090 |
| V           | 32          | 1.567   | 0.676        | 35.395              | 0 | 9.2 | 10091 |
| VI          | 32          | 1.567   | 0.806        | 33.570              | 0 | 29.6 | 10092 |
| VII         | 32          | 0.4     | 2.669        | 23.070              | 0 | 6.9 | 10089 |
| X           | 24          | 0.4     | 3.638        | 158.570             | 0 | 32.3 | 10096 |
| XI          | 16          | 0.4     | 5.246        | 91.995              | 0 | 73.4 | 10086 |
| XII         | 16          | 0.4     | 4.528        | 141.690             | 0 | 28.4 | 10087 |
| XIII        | 16          | 0.4     | 7.260        | 432.480             | 0 | 18.9 | 10088 |
We expect that the ISR will be operated with a Q split not higher than the value given in Table 1. The amplification factors indicate that noticeable coupling will be present ($\alpha, \gamma > 0.1$), if either of the tilts exceeds the following upper limits:

- for systematic tilts: $\varphi = 0.13$ mrad
- for random tilts: $\varphi = 0.37$ mrad

Similarly we can establish upper bounds for the tilt by requiring the resulting closed orbit distortion not to exceed a predetermined limit.

This limit may be set by the following argument: excluding tilted magnets the vertical closed orbit distortion is due to three equal contributions of 6.35 mm each which are added quadratically. It seems reasonable to put the same limit also on the contribution of the magnet tilts to the vertical closed orbit, thus increasing the total amount required by a factor $\sqrt{4/3} = 1.15$. 

<table>
<thead>
<tr>
<th></th>
<th>Systematic tilts</th>
<th>Random tilts</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>\alpha</td>
<td>$</td>
</tr>
<tr>
<td>$</td>
<td>\gamma</td>
<td>$</td>
</tr>
<tr>
<td>$</td>
<td>\alpha_{pv \ max}</td>
<td>$</td>
</tr>
<tr>
<td>$</td>
<td>z_{\ max}</td>
<td>$</td>
</tr>
</tbody>
</table>
Thus we have

for random tilts : \( \varphi = 0.36 \text{ mrad} \).

The distortion due to systematic tilts should be added linearly. If we limit the additional aperture to \( 0.15 \times 11 = 1.65 \text{ mm} \), as in the random case, we arrive at the following limit:

for systematic tilts: \( \varphi = 2.75 \text{ mrad} \).

By similar arguments based on a momentum spread of \( \Delta p/p = 0.04 \) between the injection orbit and the top of the stack we arrive at the following tolerances due to \( \alpha_{\text{pv max}} \):

for random tilts : \( \varphi = 1 \text{ mrad} \)
for systematic tilts: \( \varphi = 1.4 \text{ mrad} \).

We may summarize the tolerances on the tilt angles in the following table, including for comparison the upper limit for the systematic tilt and the r.m.s. value for the random tilt expected in the ISR.

### Table 7

Tolerances on magnet tilts

<table>
<thead>
<tr>
<th>Error</th>
<th>Systematic tilt [mrad]</th>
<th>Random tilt [mrad]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coupling</td>
<td>0.13</td>
<td>0.37</td>
</tr>
<tr>
<td>Closed orbit</td>
<td>2.75</td>
<td>0.36</td>
</tr>
<tr>
<td>Momentum compaction</td>
<td>1.4</td>
<td>1.0</td>
</tr>
<tr>
<td>Measurement</td>
<td>0.5</td>
<td>0.35</td>
</tr>
</tbody>
</table>
Since we cannot be sure that the tolerance for the systematic tilt is met in the ISR we consider skew quadrupoles necessary for their operation.

4.2 Proposed arrangement of skew quadrupoles

With skew quadrupoles we need no longer be concerned about the coupling, we also know that they do not have any influence on the closed orbit in first order. Therefore we also assume that the closed orbit tolerances are met.

What remains to be done is to choose an appropriate arrangement of skew quadrupoles and to make sure that the vertical momentum compaction function remains within tolerances. Table 7 shows that the tolerances on the systematic and random tilts for the momentum compaction function are about three times larger than the errors on the measurement of the tilt. Therefore, the momentum compaction function with the correction by skew quadrupoles may be a factor of up to three larger than that without correction, i.e., it should be below 90 m/rad. This fact limits the choice of the number of skew quadrupoles used for the correction. The arrangements X to XIII are excluded, though X with equal currents (Table 5) is near the boundary.

Therefore we have initially foreseen a total number of 32 skew quadrupoles for the ISR. In the present layout of the straight sections and the connecting equipment we have found 28 places for them, four locations coincide with Terwilliger quadrupole positions.

These 32 skew quadrupoles can be connected in any of the arrangements VII to IX, both with equal and different gradients. Also the arrangements X to XIII are included in the quadrupole locations chosen. Thus the ISR may be run with a smaller number of skew quadrupoles if a compromise between $\alpha_{pv \text{ max}}$ and residual coupling makes this possible, or should our assumptions about magnet tilts turn out to be too pessimistic.

4.3 Strength of skew quadrupoles

The skew quadrupoles should be capable of correcting systematic magnet tilts of 2 mrad, thus providing ample safety compared to the tilts assumed. We arrive at the following strength for the skew quadrupoles:

$$ C = \frac{dB}{dr} \times L = 0.3 \text{ T.} $$
Tolerances on the skew quadrupole gradients may be derived in the following way. The full strength of the skew quadrupoles corresponds to

$$C_{11} \approx 100 \phi,$$

where $\phi$ is the systematic tilt of the magnets. Thus an error $\Delta C/C$ in the strength of the skew quadrupoles yields an error in $C_{11}$:

$$\Delta C_{11} \approx 100 \phi \frac{\Delta C}{C},$$

which in turn corresponds to an error in the angle of the principal plane

$$\Delta \alpha \approx \frac{67 \phi}{|Q_H - Q_V|} \frac{\Delta C}{C}.$$

Thus a large $Q$ split widens the tolerances of the skew quadrupole strength. However, they are not particularly tight in practice as the following numerical example shows: $\Delta \alpha = 0.01$, $\phi = 0.5$ mrad, $|Q_H - Q_V| = 0.1$ yields

$$\Delta C/C = 3\%.$$
THE SKWORB PROGRAM

Introduction

The computer code, SKWORB, was written specifically to study the effects of magnetic tilts in a magnetic structure (such as the ISR) and to determine the correction for these effects obtainable by use of skew quadrupoles. The code computes for a set of units (magnets with specified tilts, field-free lengths, skew quadrupoles) the matrix representing the transformation of any orbit as it passes through the unit and with each successive unit computes the product matrix which represents the transformation of the orbit by all the preceding units. Since a set of units may occur periodically in the structure (such as the units of one quadrant of the ISR) the final product matrix for a set of units may be raised to any specified power obtaining the matrix which represents the transformation undergone by an orbit traversing the set that specified number of times. From the matrices thus obtained various quantities of interest with respect to the effect of tilt and degree of correction by skew quadrupoles and the required gradients of the latter are obtained.

Provision is also made in the code for starting the process with some initial transformation already experienced by the orbit. Hence results from a previous run may be extended. Matrices with and without tilt are computed for comparison.

Input for code

The first data card contains three integers (3I5 format):

NR, the number of sets of units to be done;
NM, the number of units in the set;
NK, the power to which the product matrix is to be raised.

The second data card contains a floating point number (F 10.6 format) which is $\beta_x$ estimated or previously computed. The card may also contain other information which is not read or used by the program (such as identification of problem).
The next five data cards contain row-by-row the elements of the matrix representing a transformation experienced by the orbit as it enters the set with tilt. The next five contain the elements for the same set without tilt. The format is $(5E14.6, A2)$, the description in the last field $(A2)$ is read and written by the code but not used.

The next NM (see first data card) contains five numerical quantities and an identifier which describe one-by-one the units making up the set. These are in format $(5E14.6, A2)$. The numerical quantities are in order: horizontal gradient, vertical gradient, length of unit, length of field-free space to next unit, and tilt (in radians). When the first entry (horizontal gradient normally) is zero the unit is automatically treated as a field-free unit of the specified length. When the last entry (tilt) is $\pi/4 (0.7854)$ the unit is treated as a skew quadrupole.

In the data then there is sufficient versatility to handle any set of units. The data, however, may become bulky because there must be a card for each unit. Some condensation is achieved by being able to treat a magnet and the succeeding field-free length as a single unit. The last field $(A2)$ on each data card is read and printed but is not used.

**Computation**

Unit-by-unit the code reads the data card describing the unit and computes the matrices for the unit without tilt and with specified tilt (if any) and multiplies by these, respectively, the current no tilt and tilt matrices (initially read in). For the tilt matrix, $A$, the coupling effect is represented by the amplitude of the $Z$ (vertical) motion derived from the $X$- (horizontal) components of the motion. The formulation for computing the matrices without tilt is adequately described elsewhere (SYNCH). With tilt the rotations of coordinates through the tilt angle and back are introduced prior and after the magnet transformation. The derived amplitude, $A_z(X)$, is given by

$$A_z(X) = \left[ (A_{31})^2 + (A_{32}/\beta_x)^2 \right]^{1/2}.$$
At the conclusion of the set, the tangents of the angles between the actual (due to tilt) principal planes of oscillation with the horizontal (tangent $a$) and the vertical (tangent $\gamma$) are computed. The horizontal and vertical, $\mu$, $\beta$ and $Q$ are computed and the components of the equilibrium orbit for both tilt and no tilt are computed. The maximum value of $A_z(X)$ over the set is determined.

The above constitutes a general orbit analysis. The oscillating plane angles and the maximum value of $A_z(X)$ indicate the effect of the tilt on the betatron motion. The effect of the tilt on the equilibrium orbit is indicated by its Z (third) component.

However a more significant measure of the effect of the tilt on the equilibrium orbit is the maximum absolute value attained by its Z components. After having found the equilibrium orbit for a set we can use for starting a second run $5 \times 5$ identity matrices except for the equilibrium orbit components in the first four positions in the fifth column. Then in the matrix representing each unit the fifth column will contain the components of the equilibrium orbit after that unit. The maximum value of the Z component among these is determined.

Output

Essentially all input data is output. The no-tilt (headed TOT) and tilt (headed TAT) product matrices are output for each unit. The derived amplitude in Z is output for each unit. At the end of the set, the inclination of the principal planes, the maximum derived amplitude in Z from X and the equilibrium orbits are output.
APPENDIX II

THE SYN5X5 PROGRAM

The SYN5X5 program is essentially a modification of the SYNH program written by A.A. Garren and J.W. Eusebio in Berkeley (see UCID-10153). All matrices generated are 5x5 matrices, and all matrix operations are done for 5x5 matrices. All design instructions were removed.

The following instructions are available in SYN5X5 with the same format as in SYNH, in alphabetical order: BMIS, CYA, CYB, DRF, EMIS, END, EQU, INCH, INV, MAM, OPD, PDP, REF, REM, REP, REPL, TRK, WBC, WBE, WMA. Instruction TRK now expects the initial conditions to be in the order R, R', Z, Z', ap/p, there is no plotting available.

The instructions CYA, CYB, WBC, WBE use the (R, R') and the (Z, Z') submatrices to compute the betatron functions. For the momentum compaction function the whole matrix is used.

Two instructions differ in the data format

<table>
<thead>
<tr>
<th>NAME</th>
<th>MAG</th>
<th>l</th>
<th>k</th>
<th>p</th>
<th>1 - ωx</th>
<th>γ</th>
<th>δ</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>φ</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

φ is the tilt angle.

<table>
<thead>
<tr>
<th>NAME</th>
<th>CRD</th>
<th>a₁₁</th>
<th>a₁₂</th>
<th>a₁₃</th>
<th>a₁₄</th>
<th>a₁₅</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>a₂₁</td>
<td>a₂₂</td>
<td>a₂₃</td>
<td>a₂₄</td>
<td>a₂₅</td>
</tr>
<tr>
<td></td>
<td></td>
<td>a₃₁</td>
<td>a₃₂</td>
<td>a₃₃</td>
<td>a₃₄</td>
<td>a₃₅</td>
</tr>
<tr>
<td></td>
<td></td>
<td>a₄₁</td>
<td>a₄₂</td>
<td>a₄₃</td>
<td>a₄₄</td>
<td>a₄₅</td>
</tr>
</tbody>
</table>

Thus, each row of the matrix NAME occupies one card.