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INTRODUCTION TO RENORMALONS

Guido ALTARELLI

Theoretical Physics Division, CERN
CH - 1211 Geneva 23

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1. Introduction

It has been known for a long time that the perturbative expansions in QED and QCD, after renormalization, are not convergent series [1]. This fact poses the very important conceptual problem of giving a meaning to perturbation theory. Borel has developed a general method to treat divergent series [2], but it is easy to see that in general the QED and QCD series are not even Borel summable[3-5]. Yet the Borel transform defines the result apart from ambiguities that are powers of $\exp(-1/\beta\alpha)$ where β is the first coefficient of the beta function and α is the coupling[6-11]. These terms are very small in QED and are powers of Λ^2/Q^2 in QCD, i.e. the residual ambiguities are power suppressed in the region of high momentum transfer, $Q \gg \Lambda$, where asymptotic freedom allows a perturbative approach. There has been recently a revival of interest in the subject [12-21] because one can use the large order behaviour of the perturbative expansion to get information on the errors associated with perturbative predictions, and in particular on power suppressed terms in those hard processes where the short distance operator expansion does not apply.

In the following we start from a sketchy introduction to the Borel method of treating divergent series. We then present the arguments that show that the perturbative expansions in QED and QCD do not converge. We specialize our discussion to the particularly simple and relevant case of the vacuum expectation value of the product of two electroweak currents. We summarize the singularity structure of the Borel transform in the complex plane of the associated variable. We show how the ambiguities in the resummed result are connected to this singularity structure. For QCD we then discuss in detail the connection with the short distance expansion. Finally we present a number of recent applications on power corrections to as diverse processes as τ-decay [12,13], $e^+e^- \to$ jets [14-18] and Drell–Yan lepton pair production[19,20].

2. The Borel Transform Method

Consider a divergent perturbative expansion:

$$D[a] = \sum_{n=1}^{\infty} D_n a^n = D_1 a + D_2 a^2 + \dots \; ; \tag{1}$$

one can try to give a meaning to its sum by the Borel transform method [2,7]. One defines the perturbative expansion of the Borel transform $B[b]$ of $D[a]$ by dividing the coefficients by $n!$ in the following way:

$$B[b] = \sum_{n=0}^{\infty} D_{n+1} \frac{b^n}{n!} \sim D_1 + D_2 b + D_3 \, \frac{b^2}{2} + \dots \; . \tag{2}$$

Then, formally:

$$D[a] = \int_0^{\infty} db \; e^{-b/a} B[b] \; , \tag{3}$$

in the sense that the expansion for $B[b]$ reproduces the expansion for $D[a]$ term by term. What is needed to give sense to $D[a]$ is that the integral converges (this cannot be true at all Q^2 [3,7] because of the singularities in the Q^2 plane of $D[a]$, but this problem can be neglected in our context) and that $B[b]$ has no singularities in the integration range. Consider, for example, the divergent series

$$D[a] = \sum_{n=0}^{\infty} (-1)^n \; n! \; a^{n+1} \; . \tag{4}$$

The corresponding Borel transform is given by

$$B[b] = \sum_{n=0}^{\infty} \frac{D_{n+1}}{n!} b^n = \sum_{n=0}^{\infty} (-b)^n = \frac{1}{1+b} \; , \tag{5}$$

so that $D[a]$ is given by

$$D[a] = \int_0^{\infty} db \; \frac{e^{-b/a}}{1+b} \; . \tag{6}$$

As this integral is finite and well defined (i.e. convergent with no singularities on the integration path) the method is successful and $D[a]$ has been resummed. However, note that if the sign alternation in the series of Eq. (4) is removed, the method fails. In fact from

$$D[a] = \sum_{n=0}^{\infty} n! a^{n+1} \; , \tag{7}$$

one obtains

$$B[b] = \sum_{n=0}^{\infty} \frac{d_{n+1}}{n!} b^n = \sum_{n=0}^{\infty} b^n = \frac{1}{1-b} \; , \tag{8}$$

and

$$D[a] = \int_0^{\infty} db \; \frac{e^{-b/a}}{1-b} \; . \tag{9}$$

In this case the pole at $b = 1$ is on the integration range, the integral is not well defined and the series is not Borel summable. However, we can still give a meaning to the integral by specifying a prescription to go around the pole. But the result will then depend on the chosen prescription. Fortunately, for a pole at $b = b_0$, the resulting ambiguity is of order $\exp(-b_0/a)$ so that, if this quantity happens to be small, it can be neglected. For example, if we define

$$D_{+,-}[a] = \int_0^{\infty} db \; \frac{e^{-b/a}}{1-b \pm i\epsilon} \; , \tag{10}$$

then the difference is proportional to the residue at $b = 1$:

$$D_+[a] - D_-[a] \sim \exp\left(\frac{-1}{a}\right) \; . \tag{11}$$

Similarly, exponentially small ambiguities near $a = 0$ are also obtained in the case of singularities more complicated than poles. For example, consider a case where the Borel transform shows a cut on the integration range:

$$D[a] = \int_0^\infty d\ b e^{-b/a} \log(b_0 - b) \ . \tag{12}$$

Two (among infinite) possible definitions are

$$D_{+,-}[a] = \int_0^\infty db\ e^{-b/a} \log(b_0 - b \pm i\epsilon) \ , \tag{13}$$

and the resulting difference is given by

$$D_+[a] - D_-[a] = 2\pi i \int_0^\infty db\ e^{-b/a} \theta(b - b_0) \sim \exp\left(\frac{-b_0}{a}\right) \ . \tag{14}$$

3. Dyson's Argument

A physical argument to show that the expansions in QED cannot converge was given by Dyson back in 1952 [1]. A generic observable F in QED is given in perturbation theory by an expansion in e^2:

$$F(e^2) = f_1 e^2 + f_2 e^4 + \dots \ . \tag{15}$$

If we assume that there is a finite radius of convergence in a circle around $e^2 = 0$, then we are forced to admit that there will be a convergent result also for some negative values of e^2. But the theory with negative e^2 corresponds to a force which is attractive for equal charges and repulsive for opposite charges (imagine to change the sign of e^2 in the non-relativistic Hamiltonian: the Coulomb potential term flips of sign and so does the corresponding force). For this theory it is easy to see that the vacuum is unstable. One can produce pairs of opposite charges from the vacuum by tunnel effect. In fact, the pair will repel and can be separated at large distances where one can pack together all like charges produced successively. The equal charges attract each other and the corresponding state has unbounded negative energy. Thus the perturbative series cannot converge for any negative e^2 value because otherwise one would give sense to a non-existing theory. Hence the radius of convergence must be zero.

4. Renormalon Diagrams for Vacuum Polarization Amplitudes

In order to produce a more explicit proof of the divergence of the series in QED we specialize to the case of the vacuum expectation value of two electroweak currents. We consider

$$\int d^4x\ e^{iqx} \langle 0|T(j_\mu(x) j_\nu(0))|0\rangle = (g_{\mu\nu} q^2 - q_\mu q_\nu)\Pi(q^2) \ . \tag{16}$$

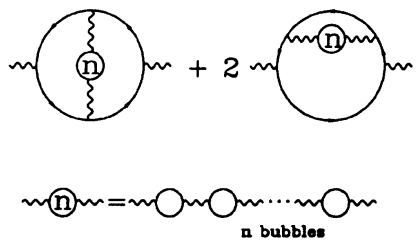


Figure 1: Renormalon diagrams for a two-point correlator

In turn the correlator $\Pi(s)$ is related to the Adler function $D(s)$, defined in such a way as to remove a constant:

$$D(s) = -s\frac{d}{ds}\Pi(s) \ . \tag{17}$$

We can directly show that the perturbative expansion for $D[a(s)]$ is divergent in QED or in QCD (where $a(s)$ is related to the running coupling). This is done by identifying a sequence of diagrams, called renormalon diagrams, shown in Fig. 1, that behave like $n!$ at large n [3-11]. At each order n in perturbation theory the bad $n!$ behaviour cannot be cancelled because the renormalon diagrams have the largest power of N_f, the number of flavours, for the given power of a. We now study the large n behaviour of the renormalon diagrams. We start with the QED case. The order n term in the expansion of the Adler function from the diagram with n blobs in Fig. 1 can be written as

$$D_n(Q^2) = \int d^4k F(k^2, kq, Q^2)\frac{a}{k^2}(\text{blob})^n \ , \tag{18}$$

where $a = a(Q^2) = \alpha(Q^2)/\pi$ and $Q^2 = q^2$. After renormalization and all angular integrations one has

$$D_n(Q^2) = \int k^2 dk^2 \Phi\left(\frac{k^2}{Q^2}\right)\frac{a}{k^2}\left[\beta a \log\left(\frac{k^2}{Q^2}\right)\right]^n \tag{19}$$

Here the leading logarithmic term of the blob has been inserted. The quantity β is the coefficient of the leading term in the QED beta function [22]:

$$\frac{da(Q^2)}{d\log(Q^2)} = \beta(Q^2) = \beta a^2(1 + \beta' a + \ldots) \quad \text{(QED)} \ , \tag{20}$$

In QED β is positive and proportional to N_f. Summing over n one obtains:

$$\begin{aligned} d(Q^2) &= \sum_{n=0}^{\infty} D_n(Q^2) = \int k^2 dk^2 \Phi\left(\frac{k^2}{Q^2}\right)\frac{1}{k^2}a(k^2) = \\ &= \int k^2 dk^2 \Phi\left(\frac{k^2}{Q^2}\right)\frac{1}{k^2}\frac{a}{1-\beta a \log\left(\frac{k^2}{Q^2}\right)} \ . \end{aligned} \tag{21}$$

We now separate the integration range at small and large k^2. We start with the infrared (IR) region from 0 to Q^2. In this region, the kernel $\Phi(k^2/Q^2)$ can be expanded in powers of its argument. We consider a generic term that behaves like $(k^2/Q^2)^h$:

$$D(Q^2) = Q^2 \int_0^{Q^2} \frac{dk^2}{k^2} \left(\frac{k^2}{Q^2}\right)^{h+1} \frac{a}{1-\beta a \log\left(\frac{k^2}{Q^2}\right)} \ . \tag{22}$$

We change variable to $t = \log Q^2/k^2$:

$$D(Q^2) = Q^2 \int_0^{\infty} dt \ e^{(-h+1)t} \frac{a}{1+\beta a t} \ . \tag{23}$$

By a further rescaling $b = at(h+1)$ we arrive at the final form:

$$D(Q^2) = \frac{Q^2}{h+1} \int_0^{\infty} db e^{-b/a} \frac{1}{1+\beta \frac{b}{h+1}} \ . \tag{24}$$

This expression is directly in the form of a Borel transform. We see that in QED the IR region leads to poles in the Borel plane at $b = -(h+1)/\beta$, on the negative axis, hence not on the integration range. Since an explicit calculation shows that $h = 1, 2, \ldots$, the poles are at $b = -2/\beta, -3/\beta, \ldots$. Note also that the poles are induced by the presence of the Landau pole in the expression of the running coupling, i.e. the pole at $\beta a \log(k^2/Q^2) = 1$ in Eq. (22).

A similar procedure can be repeated for the ultraviolet (UV) region $Q^2 < k^2 < \infty$. In this case $\Phi(k^2/Q^2)$ is expanded at large k^2 in powers of Q^2/k^2. The generic power $(Q^2/k^2)^r$, after changing variable to $t = \log k^2/Q^2$, and rescaling according to $b = (r-1)at$, leads to:

$$\begin{aligned} D[Q^2] &= Q^2 \int_{Q^2}^{\infty} \frac{dk^2}{k^2} \left(\frac{Q^2}{k^2}\right)^{r-1} \frac{a}{1-\beta a \log\left(\frac{k^2}{Q^2}\right)} = \\ &= \frac{Q^2}{r-1} \int_0^{\infty} db e^{-b/a} \frac{1}{1-\beta \frac{b}{r-1}} \ . \end{aligned} \tag{25}$$

Now it turns out that $r = 2, 3, \ldots$, so that in QED the UV region gives poles at $b = +1/\beta, +2/\beta, \ldots$. These poles are on the integration path, and they correspond to a real ambiguity in the resummation. Note that the "bad" poles, i.e. those in the integration range, correspond to the UV region, which in QED is the non-perturbative region, because the coupling increases without limit when Q^2 increases.

We now turn to the QCD case. In the non-Abelian case the blob diagrams are not sufficient by themselves to reconstruct the complete beta function. Actually the blob sequence is not even gauge-invariant. However, it is argued [3-11] that if the gauge completions of the blobs are added, the net result will simply be the complete reconstruction of the QCD beta function, which is now of the form [22]:

$$\frac{da(Q^2)}{d\log Q^2} = \beta(Q^2) = -\beta a^2(1 + \beta' a + \ldots) \qquad \text{(QCD)} \ . \tag{26}$$

The sign change in front corresponds to the fact that QCD is asymptotically free in the UV region. The coefficient β is positive and obviously different from that of QED. By repeating the same procedure as above on the blob diagrams and replacing at the end the complete QCD value of β, the final result is simply the interchange of the IR and UV regions (due to the change of sign of the leading term in the beta function). That is, in the Borel plane we now have the IR poles at $b = +2/\beta, +3/\beta, \ldots$ and the UV poles at $b = -1/\beta, -2/\beta, \ldots$. In QCD the "bad" poles arise from the IR region, which is in fact the non-perturbative one.

The pole structure that we have found in the Borel plane corresponds to a large order behaviour like:

$$D_n \sim C_k n! \left(\frac{\beta}{k}\right)^n \tag{27}$$

where, in QCD, $k = 2, 3, \ldots$ for the IR poles and $k = -1, -2, \ldots$ for the UV ones. When a number of refinements are added (the inclusion of β', logarithms, etc.) the positions of the singularities in the Borel plane remain fixed, but their nature changes, corresponding to the large-order behaviour

$$D_n \sim C_k n! n_k^\gamma \left(\frac{\beta}{k}\right)^n \quad . \tag{28}$$

We mention that there is another known source of singularities in the Borel plane from instanton contributions [3,23]. These singularities do not arise from a specific set of diagrams but rather as a collective result of an increasing multitude of diagrams at large n. In the vacuum polarization amplitudes the corresponding value of k is positive and large. Thus the corresponding ambiguities are very suppressed. Because of this, although instanton terms are conceptually important as they represent a breakdown of the short-distance expansion, we will not consider them in the following.

We have so far shown that the QED/QCD expansions are not convergent and not even Borel summable. However, in QCD, perturbation theory is only relevant at large Q^2 where $\alpha_s(Q^2)$ is small because of asymptotic freedom. But at large Q^2 the ambiguities in the Borel integral associated with the singularities at $b = k/\beta$, with $k = +2, +3\cdot$ behave as $\exp(-k/\beta a)$. But in QCD $a = 1/\beta \log Q^2/\Lambda^2$ and $\exp(-k/\beta a) = (\Lambda^2/Q^2)^k$. Thus the remaining ambiguities are power suppressed and small with respect to all individual terms in the perturbative expansion which are only logarithmically suppressed. Similarly in QED the remaining ambiguities are quite small because a is very tiny at all realistic energies and therefore $\exp(-k/\beta a)$, for $k = +1, +2, \cdots$ is exceedingly small.

5. Connection with the Operator Expansion

For the vacuum polarization amplitude under consideration an operator expansion is valid near the light cone at $x^2 = 0$ or equivalently at Q^2 large. It is thus natural to expect a correspondence between the power suppressed terms arising from the bad large-order behaviour of perturbation theory and those described by operators

of higher twist in the operator expansion. This is the issue that we discuss in the present section.

To be specific and for later reference we consider the Adler function relevant for hadronic τ decay, which is obtained from the correlator of two charged weak currents. In massless QCD with $N_f = 3$ we have [12,13]

$$\begin{aligned} D_\tau(Q^2) &= 3\Big[1 + a(Q^2) + 1.64a(Q^2)^2 + 6.37a(Q^2)^3 + \cdots + \\ &+ \frac{\text{ZERO}}{Q^2} + C_4\frac{\langle G^2_{\mu\nu}\rangle}{Q^4} + C_6\frac{\langle O_6\rangle}{Q^6} + \cdots\Big] \ . \end{aligned} \tag{29}$$

The perturbative expansion has been computed up to three loops[24]. The power suppressed terms arise from operators of increasing dimension. In massless QCD there is no gauge-invariant operator of dimension 2 that can appear in the expansion of the current product. For example $G_\mu G^\mu$, the square of the gluon field, cannot appear because of gauge invariance. This is why the coefficient of the $1/Q^2$ term was stressed in Eq. (29) to be zero. This makes the correlator of two electroweak currents rather special in that there is a gap between the leading twist and the higher twist terms. The additional terms arise at dimension 4 from the gluon condensate (by $\langle O\rangle$ we denote the vacuum expectation value of the operator O, called the O condensate). At dimension 6 we have four-fermion operators and so on. Clearly the condensates are non-perturbative quantities that arise from the IR slavery of QCD. They give a quantitative description of power suppressed terms that limit the perturbative accuracy.

We have seen that the ambiguities connected with the divergent large-n behaviour of the perturbative expansion amount to power suppressed terms. It is natural to see these terms that arise from the IR region as the perturbative way to reproduce the effect of the non-perturbative condensates. In this respect, it is interesting that the absence of an operator of dimension 2 in the operator expansion corresponds to the absence, checked by an explicit calculation[9,11], of the singularity at $b = +1/\beta$ in the Borel plane from the renormalon diagrams. Thus it is reasonable to conjecture that the power suppressed ambiguities from the divergence of the perturbative series are in one-to-one correspondence with the power suppressed terms in the operator expansion. The perturbative ambiguities can then be reabsorbed in the definition of the condensates. But it is difficult to imagine that the totality of the divergent large order behaviour is contained in the renormalon sequence. If so the whole collection of condensates would be generated by the running of α_s and by its freezing at small Q^2. Probably there are other sets of diagrams with bad large- order behaviour. But one assumes that the complete set of such diagrams will not alter the position of the singularities as obtained from the renormalon sequence. Possibly the running of α_s fixes or dominates the leading singularities in the Borel plane, and this possibility has recently been adopted by a number of authors [14-21].

In the commonly accepted wisdom about the operator expansion the coefficient functions are independent of the external states and completely determined by perturbation theory. The dependence on the external states is in the operator matrix

elements. The ambiguities of perturbation theory appear at power suppressed level in the coefficient functions. Usually it is a matter of definition at non-leading twist level to say what is the coefficient and what is the operator. But when there is a gap, as for the correlator of two currents at the level of $1/Q^2$ corrections, one must be sure to control the coefficient at the $1/Q^2$ level before considering the operator corrections at the $1/Q^4$ level [25]. We will elaborate on this issue in the next session.

6. Problems with Ultraviolet Renormalons

We have seen that the UV renormalons correspond to singularities at $b = -1/\beta, -2/\beta, \cdots$ and the IR renormalons at $+2/\beta, +3/\beta, \cdots$, (assuming that indeed there is no singularity at $b = +1/\beta$, as suggested by the blob diagrams). As a consequence, the convergence radius of the expansion for $B[b]$ near the b-origin is determined by the UV renormalon at $b = -1/\beta$, independent of the existence of the IR renormalon at $b = +1/\beta$.

Thus, the perturbative expansion for $B[b]$ can be directly used only to perform the integration up to $b = +1/\beta$. The contribution from $b = +1/\beta$ up to $b = \infty$, where the expansion is not valid, could typically lead to terms of order Λ^2/Q^2 (or even worse). For example if $B[b]$ is sufficiently well behaved at $b = +1/\beta$ and at $b = \infty$:

$$\Delta D[a] = \int_{1/\beta}^{\infty} db\ e^{-b/a} B[b] \sim aB[1/\beta]\exp(-1/\beta a) \sim aB[1/\beta]\Lambda^2/Q^2 \qquad (30)$$

where we used $a^{-1} \sim \beta \log Q^2/\Lambda^2$ and the fact that the exponential cuts away all large-b contributions so that $B[b]$ was approximated by its value near $b = +1/\beta$.

While an accuracy of order Λ^2/Q^2 is what one gets in practice from the three-loop expression of $D[a]$, it is true that, in principle, if there is no IR renormalon at $b = +1/\beta$, $D[a]$ can be better defined. In fact, as the location of the leading UV renormalon at $b = -1/\beta$ is not in the integration range, there is the possibility of defining $B[b]$ by analytic continuation up to $b = +2/\beta$. If this is realized then the remaining ambiguity, of order $(\Lambda^2/Q^2)^2$, is unavoidable because the corresponding singularity at $b = +2/\beta$ is on the real axis, so that an arbitrary procedure to go around it must be defined and the difference of two such procedures would be of that order. However, since operators of dimension 4 exist in the operator expansion, this ambiguity can be reabsorbed in the non-perturbative condensate terms[5]. We now discuss how the analytic continuation could be implemented in reality [7,12].

Starting from Eq. (3) we can make a change of variable $z = z[b]$ with inverse $b = b[z]$ and $z[0] = 0$ and $z[\infty] = 1$ (so that the interval from 0 to ∞ in b is mapped into the 0 to 1 range in z), such that the IR singularities are mapped onto the interval between $z_0 = z[2/\beta]$ and 1 and the UV singularities are pushed away at $|z| \geq 1$. Changing variable one obtains

$$D[a] = \int_0^{\infty} db\ e^{-b/a} B[b] = \int_0^1 dz |db/dz| e^{-b[z]/a} B[b[z]]\ . \qquad (31)$$

Using the expansion

$$b[z] = c_1 z + c_2 z^2 + \cdots \ , \tag{32}$$

the series

$$B[b] = D_1 + D_2 b + D_3 b^2/2 + \cdots \tag{33}$$

goes into

$$B[b[z]] = D_1 + D_2 c_1 z + \left(D_2 c_2 + \frac{D_3 c_1^2}{2} \right) z^2 + \cdots \tag{34}$$

and provides a good approximation within its radius of convergence up to $z = z_0$. Thus the improved approximation for $D[a]$ is given by

$$\begin{aligned} D[a] &\sim \int_0^{z_0} dz \left| \frac{db}{dz} \right| e^{-b[z]/a} \left\{ D_1 + D_2 c_1 z + \left(D_2 c_2 + \frac{D_3 c_1^2}{2} \right) z^2 + \cdots \right\} \\ &= \int_0^{2/\beta} db e^{-b/a} \left\{ D_1 + D_2 c_1 z[b] + \left(D_2 c_2 + \frac{D_3 c_1^2}{2} \right) z[b]^2 + \cdots \right\} \end{aligned} \tag{35}$$

where the full expression of z as a function of b is inserted in the integral. In this way, an infinite sequence of terms is added to the b expansion. For a small, the upper limit of integration can be replaced with infinity without significant effect.

One possible example is given by [7]:

$$z[b] = \frac{\sqrt{1+\beta b - 1}}{\sqrt{1+\beta b + 1|}} \rightarrow b[z] = \frac{4z}{\beta(1-z)^2} \ . \tag{36a}$$

In this case the first UV singularity is at $z = -1$, and all higher UV renormalons are on the unit circle $|z| = 1$. IR renormalons are between $z_0 = (\sqrt{3}-1)/(\sqrt{3}+1)$ and $z = 1$. In this example, $c_1 = 4/\beta$ and $c_2 = 8/\beta$. Other examples are [12]

$$z[b] = \frac{\beta b}{k + \beta b} \rightarrow b[z] = \frac{kz}{\beta(1-z)} \ , \tag{36b}$$

with $k = 1, 2$ or 3. Also in these cases the first IR renormalon at $b = 2/\beta$ becomes the closest singularity to $z = 0$, while the UV are pushed further away. Here we have $c_1 = c_2 = k/\beta$.

Note that the accelerator method is based on the mere knowledge of the position of the singularity and not on its precise form. If the nature of the singularity was known one could determine the appropriate accelerator. Clearly without this information such a method can only work if the known terms of the expansion carry enough information on the asymptotic form of the series. These statements have been quantitatively confirmed[12] by studying the problem on simple mathematical models where the true result is known. The performance of different accelerators was studied as a function of the degree of the pole singularity. In a different exercise it was also evaluated starting from a generic form for the initial terms in the expansion as a function of the coefficients of the first few terms. The results indicate that, in the

actual case of the Adler function of Eq. (29), there is little hope of improving the ambiguity from the leading UV renormalon because the first few coefficients of the actual expansion show no evidence for the asymptotic behaviour, in particular no sign alternance.

One can apply the above formalism to the problem of determining α_s from τ decay [26]. Due to the smallness of the τ mass the above problems are particularly acute. Given a fixed value of R_τ, the measured quantity, one can estimate the theoretical error on $\alpha_s(m_\tau)$ by following the variation of $\alpha_s(m_\tau)$ as a function of the procedure of acceleration, of the renormalization scale adopted, and also of different representations of R_τ (which differ by resumming or not terms of order $\beta^2 a^2 \pi^2$). The result is plotted in Fig. 2. The ambiguity from the acceleration method turns out to be a large component of the total error. In Ref. 12 we argue that the theoretical error on α_s is larger than usually taken in the experimental papers[27]. For $\delta\alpha_s(m_Z)$ we recommend a value around $\delta\alpha_s(m_Z) = \pm 0.006$.

7. Jets in e^+e^- Annihilation

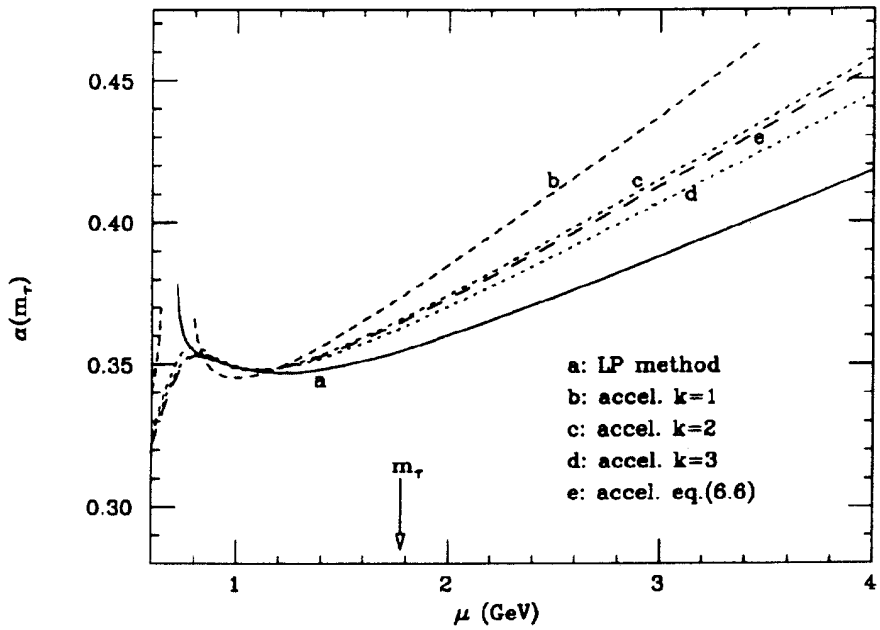
For most hard processes studied in the framework of the QCD-improved parton model there is no operator expansion. For these processes one can try to quantify the ambiguities from the perturbative expansion in order to establish, case by case, the power p of the power-suppressed corrections that behave as $(\Lambda/Q)^p$. This constructive use of the degree of divergence of the perturbative expansion has been the object of several recent studies.

Consider for example the analysis of the final state in e^+e^- annihilation. Several quantities such as jet rates or several event shape variables have been studied both theoretically and in experiment (for example at LEP) with the goal of testing QCD and of obtaining precise measurements of $\alpha_s(m_Z)$[28]. It would be very interesting to establish which quantities are the most protected from the occurrence of large power corrections, which in particular arise from hadronization.

The most rigorous approach to the perturbative expansion for the final state in e^+e^- annihilation has been discussed in Ref. 18. By taking all possible cuts of the vacuum polarization amplitude, including the iterated blobs, one obtains the corresponding absorptive part, which is connected to the production of the different final states. From the diagrams in Fig. 1 one obtains the typical three- and four-jet contributions shown in Fig. 3. In particular the four-jet term arises when the cut is through a particular blob. As before, we consider the Abelian blob model as valid also in QCD provided that the correct β coefficient is eventually replacing its Abelian counterpart. Schematically, the four-jet contribution is of the general form:

$$T = A(k^2, \phi) \left(\frac{a}{1 + \beta a \log \frac{k^2}{Q^2}} \right)^2 d\phi \frac{dk^2}{k^2} \ , \tag{37}$$

where k^2 is the virtuality of the gluon, the appropriate sign for QCD was adopted in the running coupling, ϕ stands for all other phase-space integration variables and A



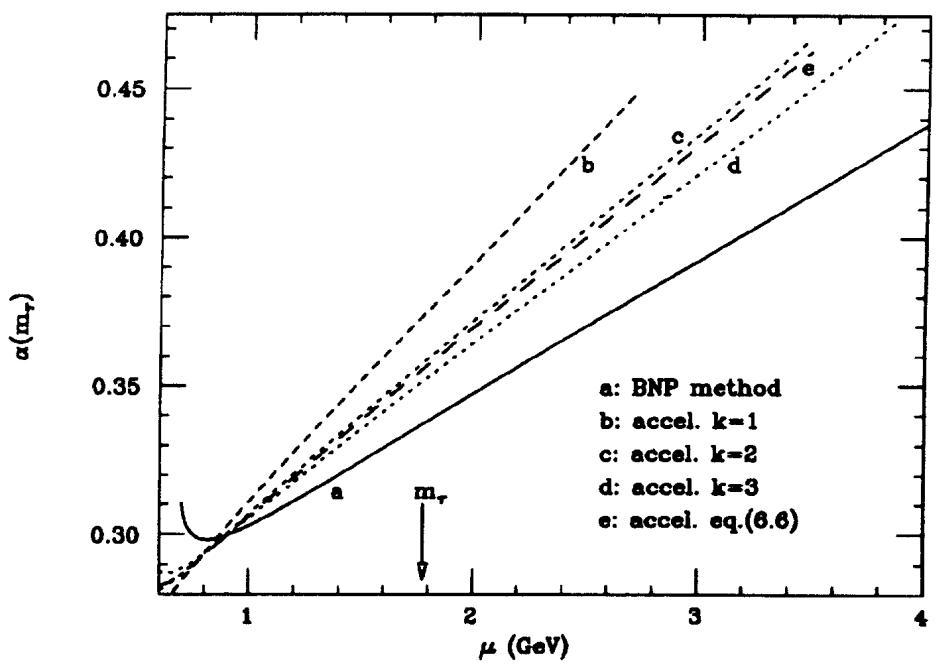


Figure 2: (a) Effect of the accelerators on the determination of $\alpha_s(m_\tau^2)$ with the Le Diberder–Pich method, for $R_\tau = 3.6$. The curves b, c and d refer to the change of variable of Eq. (36b), while e refers to Eq. (36a).
(b) As in Fig. 2a, for the Braaten, Narison, Pich method.

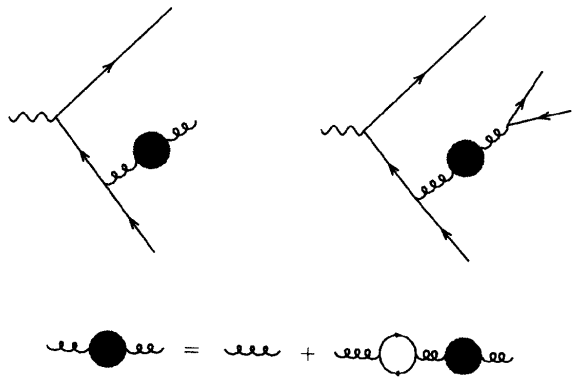


Figure 3: Dominant diagrams for e^+e^- into jets in the large n_f limit

contains the matrix element squared. For a given final-state observable S one obtains a more specific form:

$$S = \int \frac{dk^2}{k^2} \left(\frac{a}{1 + \beta a \log \frac{k^2}{Q^2}} \right)^2 G_S(k^2) \ , \tag{38}$$

with G_S being the appropriate kernel:

$$G_S(k^2) = \int d\phi A(k^2, \phi) S(k^2, \phi) \ . \tag{39}$$

Let us assume for simplicity that $G_S(0) = 0$, for the infra-red safe observable under consideration. If this was not the case, one could add and subtract $G_S(0)$. The following discussion would then apply to $G_S(k^2) - G_S(0)$, while the effect of the $G_S(0)$ term will cancel with the three-jet contribution. We now assume that at small k^2 we have

$$G_S(k^2) - G_S(0) = \left(\frac{k^2}{Q^2} \right)^{p/2} \tag{40}$$

with some positive p. By a simple change of variable, similar to what we have already seen in the previous section, we find

$$S = \int_0^{Q^2} \frac{dk^2}{k^2} \left(\frac{a}{1 + \beta a \log \frac{k^2}{Q^2}} \right)^2 \left(\frac{k^2}{Q^2} \right)^{p/2} = \int_0^\infty db e^{-b/a} \frac{a^2}{\left(1 - \frac{2\beta b}{p}\right)^2} \ . \tag{41}$$

We see that Eq. (41) corresponds to a double pole in the Borel plane at $b = p/2\beta$, and thus to a power correction like $(\Lambda^2/Q^2)^{p/2}$. In Ref. 18, Nason and Seymour make a complete analysis, starting from the Ellis–Ross–Terrano exact four jet matrix elements[29]. They compute p for each observable and find some variables that have $1/Q$ corrections while other ones have $1/Q^2$ power suppressed terms. The $1/Q$ class

of observables contains, for example, $\langle 1-T\rangle$ (T = thrust), $\langle m_H^2\rangle$ (m_H = mass of the heavy jet), $\langle\theta(O-0.1)\rangle$ (O = oblateness), the energy–energy correlation with a collinear cut, etc., while the $1/Q^2$ set includes, for example, $\langle\theta(0.8-T)\rangle, \langle\theta(m_H^2-0.1\rangle$, etc. Note that, at LEP, $1/Q \sim 1\%$, i.e. the same order as $\alpha_s(m_Z)^2$. Hence for variables that have $1/Q$ corrections it would be useless to compute the next-to-next-to-leading perturbative corrections. Actually, also for variables for which one finds $1/Q^2$ by the above method, one cannot exclude that terms of magnitude $\alpha_s(m_Z)^n/Q$ finally appear at higher orders. However, since $\alpha_s(m_Z)$ is small, these variables are still to be preferred to those that behave as $1/Q^2$ from the start.

It is interesting that most of the above results can be obtained by a simpler procedure. It is sufficient to start from the three-jet amplitude and compute it by giving a small mass λ to the gluon [14]. By looking at the small-λ behaviour of the result one can find the relevant exponent p. Alternatively one could replace a with $a(k^2)$ in a three-jet calculation [16,17]. However, it appears that these methods are not valid in general. Nason and Seymour[18] produce examples of shape variables which are identical at the three-jet level, yet differ at the four-jet level and have different values of p.

8. Drell–Yan Processes

Another application has to do with Drell–Yan processes. The cross section is a convolution of the parton densities in one hadron, those in the second hadron and the partonic cross section. In terms of moments we have the corresponding product of moments. The partonic cross-section moments have the following structure[22]:

$$\Delta_n = \exp\, 2C_F \int_0^1 dz\, \frac{z^n-1}{1-z} \int_{(1-z)Q^2/4z}^{(1-z)^2Q^2/4z} \frac{dk_T^2}{k_T^2} a(k_T^2) M(k_T^2, z)\ , \tag{42}$$

where $M(k_T^2, z)$ is the relevant matrix element squared. The effect of resumming the renormalon blobs is taken into account by the replacement $a \to a(k_T^2)$. The integration limits shown arise from the different values of $(k_T^2)_{\max}$ in deep inelastic scattering (the lower limit) and in Drell–Yan processes (the upper limit). Recall that the parton densities are defined from deep inelastic scattering. Accordingly the contribution from the integration over the deep inelastic scattering region is included in the parton densities and only the leftover is in Δ_n. Assuming that M is sufficiently regular near $z = 1, k_T^2 = 0$, one can invert the integration order and from the IR region $k_T^2 < \mu^2$, by a simple expansion, one obtains the result

$$\Delta_n = \exp 2C_F \int_0^{\mu^2} \frac{dk_T^2}{k_T^2} a(k_T^2) \int_{1-\frac{k_T}{Q}}^1 dz \frac{z^n-1}{1-z} M(0,1)\ . \tag{43}$$

On the basis of this result it was concluded[19] that corrections of order $1/Q$ are to be expected in Drell–Yan processes. However, it was shown[20] that this result is not true and the corrections only arise at order $1/Q^2$. The bug in the above argument is that

the coefficients of the expansion in k_T^2 of the quantity M are singular at $z = 1$, as shown in Ref. 20 by an explicit calculation. As a consequence one cannot expand in k_T^2 before the z integration. The $1/Q$ correction arising from the phase-space restriction is actually cancelled by the effect of M near $z = 1$, due to soft gluons. In conclusion the corrections to totally inclusive lepton pair production behave as $1/Q^2$.

9. Outlook

In my opinion the main open problems in this subject are to better understand the relation between the divergences of the perturbative expansion and the power-suppressed corrections. For example it would be interesting to identify other sequels of diagrams with bad large-order behaviour in order to show that the freezing of α_s is not the only source of power-suppressed terms. In any case one should understand to what extent the freezing of α_s is important. Does it at least determine the leading power-suppressed terms? More in general are there other sources, beyond instantons, that lead to a breakdown of the light cone expansion? We hope that some of these questions can be answered in the near future.

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