Estimate of $\mathcal{B}(\bar{B} \to X_s \gamma)$ at $O(\alpha_s^2)$

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Combining our results for various $O(\alpha_s^2)$ corrections to the weak radiative $B$-meson decay, we are able to present the first estimate of the branching ratio at the next-to-next-to-leading order in QCD. We find $\mathcal{B}(\bar{B} \to X_s \gamma) = (3.15 \pm 0.23) \times 10^{-4}$ for $E_\gamma > 1.6$ GeV in the $B$-meson rest frame. The four types of uncertainties: nonperturbative (5%), parametric (3%), higher-order (3%), and $m_t$-interpolation ambiguity (3%) have been added in quadrature to obtain the total error.

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The inclusive radiative $B$-meson decay provides important constraints on the minimal supersymmetric standard model and many other theories of new physics at the electroweak scale. The power of such constraints depends on the accuracy of both the experiments and the standard model (SM) calculations. The latest measurements by Belle and BABAR are reported in Refs. [1,2]. The world average performed by the Heavy Flavor Averaging Group [3] for $E_\gamma > 1.6$ GeV reads

$$\mathcal{B}(\bar{B} \to X_s \gamma) = (3.55 \pm 0.24^{+0.09}_{-0.10} \pm 0.03) \times 10^{-4}. \quad (1)$$

The combined error in the above result is of the same size as the expected $O(\alpha_s^2)$ next-to-next-to-leading order (NNLO) QCD corrections to the perturbative decay width $\Gamma(b \to X_{\text{parton}}^s \gamma)$, and larger than the known nonperturbative corrections to the relation $\Gamma(\bar{B} \to X_s \gamma) \approx \Gamma(b \to X_{\text{parton}}^s \gamma)$ [4–6]. Thus, calculating the SM prediction for the $b$-quark decay rate at the NNLO is necessary for taking full advantage of the measurements.

Evaluating the $O(\alpha_s^2)$ corrections to $\mathcal{B}(b \to X_{\text{parton}}^s \gamma)$ is a very involved task because hundreds of three-loop on-shell and thousands of four-loop tadpole Feynman diagrams need to be computed. In a series of papers [7–14], we have presented partial contributions to this enterprise. The purpose of the present Letter is to combine all the existing results and obtain the first estimate of the branching ratio at the NNLO. We call it an estimate rather than a prediction because some of the numerically important contributions have been found using an interpolation in

FIG. 1. Sample LO diagram for the $b \to s \gamma$ transition.
Coupling constants at these vertices (Wilson coefficients) are first evaluated at the electroweak renormalization scale $\mu_0 \sim m_t, M_W$ by solving the so-called matching conditions. Next, they are evolved down to the low-energy scale $\mu_b \sim m_b$ according to the effective theory renormalization group equations (RGE). The RGE are governed by the operator mixing under renormalization. Finally, one computes the matrix elements of the operators, which in our case amounts to calculating on-shell diagrams with single insertions of the effective theory vertices.

A summary of the $\bar{B} \to X_s \gamma$ calculation status before the beginning of our project can be found, e.g., in Ref. [15]. At the NNLO level, the dipole and the four-quark operators need to be matched up to three and two loops, respectively. Renormalization constants up to four loops must be found for $b \to s \gamma$ and $b \to s g$ diagrams with four-quark operator insertions, while three-loop mixing is sufficient in the remaining cases. Two-loop matrix elements of the dipole operators and three-loop matrix elements of the four-quark operators must be evaluated in the last step.

Three-loop dipole operator matching was found in Ref. [8]. The necessary three-loop mixing was calculated in Ref. [9]. The four-loop mixing was evaluated in Ref. [13]. Two-loop matrix element of the photonic dipole operator together with the corresponding bremsstrahlung was found in Refs. [10,11] and recently confirmed in Ref. [12]. Three-loop matrix elements of the four-quark operators were found in Ref. [7] within the so-called large-$\beta_0$ approximation. A calculation that goes beyond this approximation by employing an interpolation in the charm quark mass $m_c$ has just been completed in Ref. [14].

With all these results at hand, we are ready to present the first estimate of the $\bar{B} \to X_s \gamma$ branching ratio at $O(\alpha_s^3)$. It reads [16]

$$B(\bar{B} \to X_s \gamma) = (3.15 \pm 0.23) \times 10^{-4},$$

for $E_\gamma > 1.6$ GeV in the $\bar{B}$-meson rest frame. The four types of uncertainties: nonperturbative (5%), parametric (3%), higher-order (3%), and $m_c$-interpolation ambiguity (3%) have been added in quadrature in Eq. (2).

The central value in Eq. (2) was obtained for $\mu_0 = 160$ GeV, $\mu_b = 2.5$ GeV, and $\mu_c = 1.5$ GeV. The latter quantity stands for the charm mass $\overline{\text{MS}}$ renormalization scale that is allowed to be different from $\mu_b$. The branching ratio dependence on each of the three scales is shown in Fig. 2. Once one of them is varied, the remaining two are fixed at the values that have been mentioned above. The reduction of the renormalization scale dependence at the NNLO is clearly seen. The most pronounced effect occurs for $\mu_c$, that was the main source of uncertainty at the NLO. (The LO results are $m_c$ and thus $\mu_c$ independent.) The current uncertainty of $\pm 3\%$ due to higher-order [$O(\alpha_s^5)$] effects is estimated from the NNLO curves in Fig. 2.

The reference value of $\mu_b = 2.5$ GeV that we have chosen is roughly twice smaller than in the previous LO and NLO analyses. Given the stability of the NNLO result for large values of $\mu_b$, we do not underestimate any uncertainty from that region. Furthermore, because the center-of-mass energy $m_{\bar{B}} \approx 5.3$ GeV gets distributed among various partons, the reference value of $\mu_b = 2.5$ GeV seems reasonable. Lower values of $\mu_b$ have an advantage of making $\mu_c$ stabilization more efficient because the NNLO logarithm that compensates $\mu_c$ dependence of the NLO amplitude comes multiplied by $\alpha_s(\mu_b)$.

The $\pm 3\%$ uncertainty that is assigned to the $m_c$-interpolation ambiguity has been estimated studying by how much the NNLO branching ratio depends on various interpolation assumptions. More details on this point and other elements of the phenomenological analysis (including the input parameters) can be found in Ref. [14].

As far as the parametric uncertainties are concerned, the dominant ones come from $\alpha_s(M_Z)$ ($\pm 2.0\%$) and the measured semileptonic branching ratio $B(\bar{B} \to X_s e \nu)$ ($\pm 1.6\%$) to which we normalize. The third-to-largest uncertainty ($\pm 1.1\%$) is due to the correlated errors in $m_c(m_c)$ and the
The nonperturbative uncertainty in Eq. (2) is due to
matrix elements of the four-quark operators in the presence
of one gluon that is not soft ($Q^2 \sim m_g^2$, $m_b\Lambda$, where $\Lambda \sim \Lambda_{QCD}$). Unknown
nonperturbative corrections to them scale like $\alpha_s\Lambda/m_b$ in the limit $m_c \ll m_b/2$ and like
$\alpha_s^2\Lambda^2/m_b^2$ in the limit $m_c \gg m_b/2$. Because $m_c < m_b/2$ in reality, $\alpha_s\Lambda/m_b$ should be considered as the quantity
that sets the size of such effects. Consequently, a $\pm 5\%$
nonperturbative uncertainty has been assigned to the result
in Eq. (2). This is the dominant uncertainty at present.
Thus, a detailed analysis of such effects would be more
than welcome. So far, no published results on this issue
exist. Even lacking a trustworthy method for calculating
such effects, it might be possible to put rough upper bounds
on them that could supersede the current guess-estimate of
$\pm 5\%$. Nonperturbative corrections to inclusive $B \to X_d\gamma$
decays that scale like $\Lambda/m_b$ may arise when the $b$-quark
annihilation vertex does not coincide with the hard photon
emission vertex; see, e.g., Ref. [6] or comments on $B \to X_d\gamma$
in Sec. 2 of Ref. [5].

The NNLO central value in Eq. (2) differs from some of
the previous NLO predictions by between 1 and 2 error
bars of the NLO results. Because those error bars were
obtained by adding various theoretical uncertainties in
quadrature, such a shift is not improbable, similarly to
shifts by less than $2\sigma$ in experimental results. The shift
from the NLO to the NNLO level diminishes with lowering
the value of $\mu_c$, which has motivated us to use the relatively
low $\mu_c = 1.5$ GeV as a reference value here.

The NNLO results turn out to be only marginally dependant on whether one follows (or not) the approach of
Ref. [18] where the top-quark contribution to the decay amplitude was calculated separately and rescaled by quark
mass ratios to improve convergence of the perturbation series. Although the top contribution alone indeed behaves
better also at the NNLO level when such an approach is used, the charm quark contribution (to which no rescaling
has been applied in Ref. [18]) does not turn out to be particularly stable beyond the NLO. Consequently, in the
derivation of Eq. (2) and Fig. 2, we have used the simpler method of treating charm and top sectors together.

Our result in Eq. (2) has been obtained under the assumption that the photonic dipole operator contribution to the
integrated $E_\gamma$ spectrum below 1.6 GeV is well approxi-
mated by a fixed-order perturbative calculation (see Note
added). For lower values of the photon energy cut, the
following numerical fit can be used:

$$\frac{\mathcal{B}(E_\gamma > E_0)}{\mathcal{B}(E_\gamma > 1.6 \text{ GeV})}_{\text{fixed order}} = 1 + 0.15x - 0.14x^2,$$

where $x = 1 - E_0/(1.6 \text{ GeV})$. This formula coincides with our NNLO results up to $\pm 0.1\%$ for $E_0 \in [1.0, 1.6] \text{ GeV}$. The error is practically $E_0$-independent in this range.

In the remainder of this Letter, we shall update the $B \to X_d\gamma$ constraints on the charged Higgs boson mass in the
two-Higgs-doublet-model II (THDM II) [19]. The solid
lines in Fig. 3 show the dependence of $\mathcal{B}(B \to X_d\gamma)$ on
this mass when the ratio of the two vacuum expectation
values, $\tan\beta$, is equal to 2. The dashed and dotted lines
show the SM (NNLO) and the experimental results, respectively. In each case, the middle line is the central value,
while the other two lines indicate uncertainties that one
obtains by adding all the errors in quadrature.

In our THDM calculation, matching of the Wilson coefficients at the electroweak scale is complete up to the
NLO [20], but the NNLO terms contain only the SM contributions (the THDM ones remain unknown). In consequence, the higher-order uncertainty becomes somewhat larger. This effect is estimated by varying the matching scale $\mu_q$ from half to twice its central value. It does not exceed $\pm 1\%$ for the $M_{H^+}$ range in Fig. 3.

Even though the experimental result is above the SM
one, the lower bound on $M_{H^+}$ for a generic value of $\tan\beta$
remains stronger than what one can derive from any other
currently available measurement. If all the uncertainties
are treated as Gaussian and combined in quadrature, the
95% (99%) C.L. bound amounts to around 295 (230) GeV.
It is found for $\tan\beta = \infty$ but stays practically constant
down to $\tan\beta \approx 2$. For smaller $\tan\beta$, the branching ratio
and the bound on $M_{H^+}$ increase.

The contour plot in Fig. 4 shows the dependence of the
$M_{H^+}$ bound on the experimental central value and error.
The current experimental result (1) is indicated by the
black square. Consequences of the future upgrades in the
measurements will easily be read out from the plot, so long as no progress on the theoretical side is made. Of course, the derived bounds should be considered illustrative only because they depend very much on the theory uncertainties that have no statistical interpretation.

To conclude, we have provided the first estimate of $\mathcal{B}(B \to X_s \gamma)$ at $O(\alpha_s^2)$. The inclusion of the NNLO QCD corrections leads to a significant suppression of the branching ratio renormalization scale dependence that has been the main source of uncertainty at the NLO. The central value is shifted downward with respect to all the previously published NLO results. It is now about 1 or lower than the experimental average (1). The dominant theoretical uncertainty is currently due to the unknown $O(\alpha_s \Lambda/m_b)$ non-perturbative effects. In the two-Higgs-doublet model II, the experimental results favor a charged Higgs boson mass of around 650 GeV. The 95\% C.L. bound for this mass amounts to around 295 GeV if all the uncertainties are treated as Gaussian.

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Note added. —Recently, our results from Eqs. (2) and (4) were combined in Ref. [21] with perturbative cutoff-related corrections that go beyond a fixed-order calculation [21,22]. Because these corrections for $E_0 \leq 1.6$ GeV do not exceed our higher-order uncertainty of $\pm 3\%$, we postpone their consideration to a future upgrade of the phenomenological analysis, where other contributions of potentially the same size are going to be included, too (see Sec. I of Ref. [23]).

[16] The small ($\sim 0.35\%$) correction from the four-loop $b \to s g$ mixing diagrams is not included in our numerical results.